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On $*$ −Skew $*$ −Armendariz $*$ −Rings

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Abstract

A ring R is called $\alpha - * -$skew $* -$Armendariz $* -$rings if whenever the polynomials $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} a_j\, x^j \in R[x,\alpha]$ satisfy $p(x)q(x) = p(x)q^*(x) = 0$, then $a_i \alpha^j(b_j) = 0$ for all i,j (consequently $a_i \alpha^j(b_j^*) = 0$), which is a proper generalization of reduced $* -$rings. We study the condition for $\alpha - * -$skew $* -$Armendariz $* -$rings to be reduced. In addition, we discuss many properties of $\alpha - * -$skew $* -$Armendariz $* -$ring. Also, we give the relationship between the Baerness of a $* -$ring R. Finally, we generalize the property of $\alpha - * -$skew *-Armendariz to some know extensions.

Key Words: $* -$Armendariz $* -$rings, $* -$skew polynomial $* -$rings, rigid $* -$rings, $* -$Baer $* -$rings.

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حول تمديد خاصية ارماندريز الانحرافية للحلقات الالتفافية

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الملخص

الحلقات الإلتفافية تسمي تمديد حلقات أرماندريز الإلتفافية الأنحرافية عندما تكون عندنا أثنان من متعددة الحدود $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} a_j x^j \in R[x,\alpha]$ تحقق $p(x)q(x) = p(x)q^*(x) = 0$، فأن $a_i \alpha^j(b_j) = 0$ لكل i،j وتتبعها $a_i \alpha^j(b_j^*) = 0$، الذي يكون تعميم للحلقات المختزلة وندرس الشرط علي تمديد حلقات أرماندريز الإلتفافية الأنحرافية لتكون حلقات مختزلة، ونحن أولا نقوم بمناقشة الكثير من خصائص تمديد حلقات أرماندريز الإلتفافية الأنحرافية بالأضافة الي ذلك دراسة العلاقة بينها وبين حلقات بيير الألتفافية، وأخيرا نعمم خاصية تمديد حلقات أرماندريز الإلتفافية الأنحرافية الي بعض التمديدات المعروفة.

1.Introduction

By a ring we always mean an associative ring with identity. A ring R is said to be $*-$ring if on R there is defined an involution $*$. $*-$rings are objects of the category of rings with involution with morphisms also preserving involution.
Therefore the consistent way of investigating $*-$rings is to study them within this category, as done in a series of papers (for instance [1],[2],[3],[4],[5],[6],[7] and [10]). The purpose of this paper is to study $\alpha - * -$skew $*-$Armendariz $*-$rings within its category. A self adjoint idempotent element e (that is $e^* = e = e^2$) is called a projection. A $*-$ring R is said to be Abelian ($-$Abelian) if every idempotent (projection) of R is central. We denote the set of all projections of R by $B_*(R)$. The study of the Armendariz rings which is related to polynomial rings, was initiated by Armendariz [7] and Rege and Chhawchharia [6]. A ring R is called Armendariz if

whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m, g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. (The converse is obviously true). Recall from [10], an element a of R is said to be $*-$nilpotent if $(aa^*)^n = 0$ and $a^m = 0$ for some positive integers n and m. A $*-$ring R is called reduced ($*-$reduced) if it has no nonzero nilpotent ($*-$nilpotent) elements. Reduced rings are Armendariz by [7, Lemma1]. Birkenmeiera et al.[3], defined a *-ring R as a Baer $*-$ring if the right annihilator of every nonempty subset of R is generated, as a right ideal, by a projection. In [10], a generalization of a Baer $*-$ring is given which is consistent with the category of involution rings that is a $*-$Baer $*-$ring. A $*-$ring R is said to be a $*-$Baer $*-$ring if the $*-$right annihilator of every nonempty subset A of R is a principal $*-$biideal generated by a projection: that is $r_*(A) = e$ According to Krempa [5], an endomorphism α of a ring R is called to be rigid ($*-$rigid) if $a\alpha(a) = 0$ $(ab^2 = abb = 0)$ implies $a = 0, (ab = 0)$
for $a, b \in R$. We call a ring R $\alpha-$rigid ($\alpha-*-$rigid) if there exists a rigid ($*-$rigid) endomorphism ($*-$endomorphism) α of R. Note that any rigid endomorphism of a ring is a monomorphism and α -rigid rings are reduced rings by Hong et al. [2, Proposition 5]. Properties of α -rigid rings have been studied in Krempa [4],
Hong et al. [2], and Hirano [4] Recall that for a ring R with a ring endomorphism $\alpha: R \longrightarrow R$, a skew polynomial ring (also called an Ore extension of endomorphism type) $R[x, \alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $x\mathrm{r} = \alpha(\mathrm{r})x$ for all $r \in R$. Throughout this paper, The natural numbers, the integers, the rational numbers, the real numbers and the complex numbers will be denoted by $\mathrm{N}, \mathrm{Z}, Q, \mathrm{R}$ and C, respectively. $M_n(\mathrm{R})$ will denote the full matrix ring of all $n \times n$ matrices over the ring R, while $T(\mathrm{R})$ ($T_{nE}(\mathrm{R})$) will denote the $n \times n$ upper triangular matrix ring (with equal diagonal elements) over R. Moreover, A $*-$endomorphism on a $*-$ring R is a homomorphism $\alpha: R \longrightarrow R$ satisfy $\alpha(a^*) = \alpha(a)^*$ for all $a \in R$. Here, we consider only $*-$endomorphisms that are nonzero and nonidentity unless otherwise specified. The $*-$endomorphism α on $*-$ring R can be

extended to $\tilde{\alpha}$ on $T_{nE}(\mathrm{R})$ (resp., $M_n(\mathrm{R})$) and $R[x]$ by $\tilde{\alpha}\left(\left(a_{ij}\right)\right) = \left(\alpha\left(a_{ij}\right)\right)$ and $\alpha\left(\sum_{i=0}^{m} a_i x^i\right) = \sum_{i=0}^{m} \alpha(a_i) x^i$ for all i, j, respectively.

Furthermore, for a commutative ring$R$, the involution $\diamond$ defined on $T_{nE}(\mathrm{R})$ for $n > 2$ is given by replacing each entry by it involutive image and fixing the two diagonals considering the diagonal right upper = left lower as symmetric ones and interchanging the symmetric elements about it. For $n = 2$ (trivial extension$T(R, R)$, the involution $\diamond$ is the adjoint involution.

2. α −∗ −skew ∗ −Armendariz ∗ −rings

In this section, α −∗ −skew ∗ −Armendariz ∗ −rings are introduced as a generalization for ∗ −Armendariz, α −∗ −Armendariz and α −rigid ∗ −rings.

Definition. Let α be an ∗ −endomorphism on a ∗ −ring R. R is called α −∗ −skew ∗ −Armendariz ∗ −rings if whenever the polynomials $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} a_j\, x^j \in R[x, \alpha]$ satisfy $p(x)q(x) = p(x)q^*(x) = 0$, then $a_i \alpha^j(b_j) = 0$ for all i, j (consequently $a_i \alpha^j(b_j^*) = 0$).

It can be easily checked that if R is an ∗ −Armendariz ∗ −ring then it is an I_R −∗ −skew ∗ −Armendariz ∗ −ring, where I_R − is an identity ∗ −endomorphism of
R. Note that every subring of a α −∗ −skew ∗ −Armendariz ∗ −ring is α −∗ −skew ∗ −Armendariz.

Each, α −skew− Armendariz and Armendariz ∗ −rings are clearly α −∗ −skew ∗ −Armendariz, but the converse is not true by the following example.

Example 1. The ∗ −ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ over a field F, with the adjoint involution and a ∗ −endomorphism $\alpha: R \longrightarrow R$ defined by

$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$, since for any projection $e \in R, \alpha(e) = e$ and so R is $\alpha -* -$Armendariz, hence R is $\alpha -* -$skew $* -$Armendariz. Moreover, R is not $\alpha -$skew$-$ Armendariz **[1, Example 12]**. Furthermore, R is not Armendariz.

The following example shows that there exists a $* -$endomorphism $\alpha -$ of a $* -$Armendariz $* -$ ring R such that R is not $\alpha -* -$skew $* -$Armendariz.

Example 2. The *-ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with the changeless involution $*$ defined as $(a, b)^* = (a^*, b^*)$ and *-automorphism $\alpha: R \longrightarrow R$ given by $\alpha((a, b)) = (b, a)$ is a commutative reduced ring. Thus it is $* -$Armendariz. Moreover, R is not $\alpha -* -$skew $* -$Armendariz, since the skew polynomial $p(x) = (1, 0)x \in R[x, \alpha]$, satisfies $p^2 = pp^* = 0$, while $p \neq 0$.

From **[1, Corollary 4]**,each $\alpha -$rigid $* -$ring is $\alpha -* -$skew $* -$Armendariz, but the converse is not true as shown by the following example:

Example 3. The $* -$ring $R = T(\mathbb{Z}, \mathbb{Q})$ with adjoint involution is $\alpha -* -$ Armendariz **[1, Example 1]** and so $\alpha -* -$skew $* -$Armendariz. Moreover $T(\mathbb{Z}, \mathbb{Q})$ is not $\alpha -$rigid, since the matrix $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ satisfies $A\alpha(A) = 0$, while $A \neq 0$. Also, from **[1, Example 5]** R is $\alpha -$skew$-$ Armendariz $* -$ring and so $\alpha -* -$skew $* -$Armendariz, but is not $\alpha -$rigid.

However, there is no clear connection between $\alpha -* -$rigid and $\alpha -* -$skew $* -$Armendariz $* -$rings. By the way, the **(Example 3)** declare that there exists an $\alpha -* -$skew $* -$Armendariz which is not $\alpha -* -$rigid, such that $A\alpha(A) = A\alpha(A^*) = 0$, while $A \neq 0$.

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Adding the condition a reduced to $\alpha - * -$skew $* -$Armendariz $* -$ring makes it $\alpha - * -$rigid.

Proposition 1. If R is a reduced $\alpha - * -$skew $* -$Armendariz $* -$ring, then R is $\alpha - * -$rigid.

Proof. Assume the conclusion is false. Then there exists an element $a \in R$ such that $a \neq 0$ and $a\alpha(a) = a\alpha(a^*) = 0$. We certainly have $\alpha(a)\alpha^2(a) = \alpha(a\alpha(a)) = \alpha(0) = 0$, $\alpha(a)\alpha^2(a^*) = \alpha(a\alpha(a^*)) = \alpha(0) = 0$. Also, we have $\alpha(a)a = 0$. In fact $(\alpha(a)a)^2 = \alpha(a)(a\alpha(a))a = 0$. Since R is reduced, so $\alpha(a)a = 0$. Since $a \neq 0$, α is a monomorphism, and R is reduced it follows that $\alpha(a) \neq 0$ and $(\alpha(a))^2 \neq 0$. For any $p(x) = a_0 + a_1x$, and $q(x) = b_0 + b_1x$ in $R[x, \alpha]$, we have:-

$p(x)q(x) = a_0 b_0 + (a_0 b_1 + a_1\alpha(b_0))x + a_1\alpha(b_1)x^2$ and

$p(x)q^*(x) = a_0 b_0^* + (a_0 b_1^* + a_1\alpha(b_0^*))x + a_1\alpha(b_1^*)x^2$.

Especially tak $a_0 = \alpha(a)$, $a_1 = \alpha(a)$, $b_0 = a$, $b_1 = -\alpha(a)$. Then

$p(x)q(x) = \alpha(a)\, a + (-\alpha(a)\alpha(a) + \alpha(a)\alpha(a))x + (-\alpha(a)\alpha^2(a))x^2 = 0$ and $p(x)q^*(x) = \alpha(a)\, a^* + (-\alpha(a)\alpha(a^*) + \alpha(a)\alpha(a^*))x + (-\alpha(a)\alpha^2(a^*))x^2$ in $R[x, \alpha]$. But $a_0 b_1 = -\alpha^2(a) \neq 0, a_0 b_1^* = -\alpha(a)\alpha(a^*) \neq 0$. This shows that R is not $\alpha - * -$skew $* -$Armendariz, a contradiction. Hence R is $\alpha - * -$rigid.

Since a reduced $* -$ring is $* -$reduced, we have the following corollaries.

Corollary 1. If the Ore extension $R[x, \alpha]$ is a reduced $* -$ring, then $* -$ring R is _ $\alpha - * -$rigid.

The **(Example 2)** confirms previous corollaries, such that the $* -$ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with the changeless involution $*$ and $* -$automorphism $\alpha: R \longrightarrow R$ is not ($\alpha -$ rigid) $\alpha - * -$rigid, since the nonzero element $A = (1,0)$ satisfies $A\alpha(A) = A\alpha(A^*) = 0$, while $A \neq 0$. Moreover, $R[x,\alpha]$ is not (reduced) $*-$reduced, since the skew polynomial $p(x) = (1,0)x \in R[x,\alpha]$, satisfies $p^2 = pp^* = 0$, while $p \neq 0$.

Since each reduced $*-$ring is $*-$Armendariz **[8, Proposition 1]**, we have the following corollary

Corollary 2. Every reduced $*-$ring is $\alpha - * -$skew $*-$Armendariz.

The converse of the previous corollary is not true as clear from the following example:

Example 4. The $*-$ring $R = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ with adjoint involution $*-$ and $*-$endomorphism $\alpha = *$ is $*-$Armendariz **[8, Example 1]** and so $\alpha - * -$skew $*-$Armendariz. Moreover, R is not reduced since the nonzero matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $A^2 = 0$.

One can easily show that the class of $\alpha - * -$skew $*-$Armendariz *-rings is closed under finite subdirect sums (with changeless involution) and under taking $-$subrings.

Proposition 2. The class of $\alpha - * -$skew $*-$Armendariz $*-$rings is closed under finite subdirect sums and under taking $*-$subrings.

Proposition 3. Let R be a commutative $\alpha-$rigid $*-$ring and α be a $*-$endomorphism on R, then the $\diamond-$ring $T_{3E}(R)$, with adjoint involution $\diamond$, is $\tilde{\alpha} - \diamond -$skew $\diamond -$Armendariz $\diamond -$rings.

In case of trivial extension $T(R,R)$ with adjoint involution R given by $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^{\diamond} = \begin{pmatrix} a^* & -b \\ 0 & a^* \end{pmatrix}$, then from **(Proposition 3)** we get the following result.

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Corollary 4. Let R be a commutative α −rigid $*$ −ring and α − be a $*$ −endomorphism of R, then $T(R,R)$ with adjoint involution $\diamond$ is $\tilde{\alpha}-\diamond$ −skew $\diamond$ −Armendariz $\diamond$ −rings.

The α −rigid condition in **(Corollary 4)** is essential and cannot be replaced by the reduced condition according to the following example:

Example 5. From **(Example 2)**, the *-ring $R=\mathbb{Z}_2\oplus\mathbb{Z}_2$ is not α −rigid such that nonzero element $A=(1,0)$ satisfies $A\alpha(A)=0$, while $A\neq 0$ and is reduced. Moreover, the $\diamond$ −ring $T(R,R)$ is not $\tilde{\alpha}-\diamond$ −skew $\diamond$ −Armendariz $\diamond$ −rings,
since the polynomials
$p(x)=\begin{pmatrix}(1,0) & (0,0)\\ (0,0) & (1,0)\end{pmatrix}+\begin{pmatrix}(1,0) & (0,0)\\ (0,0) & (1,0)\end{pmatrix}x, q(x)=$
$\begin{pmatrix}(0,1) & (0,0)\\ (0,0) & (0,1)\end{pmatrix}+\begin{pmatrix}(1,0) & (0,0)\\ (0,0) & (1,0)\end{pmatrix}x\in T(R,R)[x,\tilde{\alpha}]$,
satisfy $p(x)q(x)=p(x)q^{\diamond}(x)=0$,
While
$\begin{pmatrix}(1,0) & (0,0)\\ (0,0) & (1,0)\end{pmatrix}\tilde{\alpha}\left(\begin{pmatrix}(0,1) & (0,0)\\ (0,0) & (0,1)\end{pmatrix}\right)=$
$\begin{pmatrix}(1,0) & (0,0)\\ (0,0) & (1,0)\end{pmatrix}\begin{pmatrix}(1,0) & (0,0)\\ (0,0) & (1,0)\end{pmatrix}\neq 0.$

Lemma 1. Let R be an $\alpha-*$ −skew $*$ −Armendariz $*$ −ring. If $e^2=e\in R[x,\alpha]$, where $e=e_0+e_1x+\cdots+e_nx^n$ for all projections, then $e=e_0$.

Proof. Since $e(1-e)=0=(1-e)e$, $e(1-e^*)=(1-e^*)e$, we have $(e_0+e_1x+\cdots+e_nx^n)\big((1-e_0)-e_1x-\cdots-e_nx^n\big)=0$, $(e_0+e_1x+\cdots+e_nx^n)((1-e_0)^*-e_1{}^*x-\cdots-e_n{}^*x^n)=0$ and $\big((1-e_0)-e_1x-\cdots-e_nx^n\big)(e_0+e_1x+\cdots+e_nx^n)=0$, $((1-e_0)^*-e_1{}^*x-\cdots-e_n{}^*x^n)\,(e_0+e_1x+\cdots+e_nx^n)=0$. Since R is an $\alpha-*$ −skew $*$ −Armendariz $*$ −ring, $e_0(1-e_0)=0$, $e_0e_i=0$ and $(1-e_0)e_i=0$ for $1\le i\le n$. Thus $e_i=0$ for $1\le i\le n$, and so $e=e_0=e_0^2$.

The following results partially include the results of Aburawash and Elgamudi **[8]**.

Proposition 4. Let R be an $\alpha -* -$skew $* -$Armendariz $* -$ring. Then $\alpha(e) = e$ for any $e^2 = e = ee^* \in R$ if $R[x,\alpha]$ is a $* -$ belian.

Proof. Suppose that $R[x,\alpha]$ is a $* -$ belian and $e^2 = e = ee^*$. Then e is central
and so $ex = xe = \alpha(e)$. Thus $e = \alpha(e)$.

Theorem 1. Assume that α is a $* -$utomorphism of a $* -$ring R with $\alpha(e) = e$ for any $e^2 = e = ee^* \in R$. If R is an $\alpha -* -$skew $* -$Armendariz $* -$ring, then R is a $* -$ Baer $* -$ring if and only if $R[x,\alpha]$ is a $* -$Baer $* -$ring.

Proof. Assume that R is a *-Baer. Let A be a nonempty subset of $R[x,\alpha]$ and B be the set of all coefficients of elements of A then B is a nonempty subset of R and so $r_*(B) = eRe$ for some projection $e \in R$. Since $e \in r_{*R[x,\alpha]}(A)$
we get $eR[x,\alpha]e \subseteq r_{*R[x,\alpha]}(A)$. Now let $0 \neq g = b_0 + b_1x + \_\_\_ + b_mx^m \in r_{*R[x,\alpha]}(A)$. Then $Ag = Ag^* = 0$ and hence $fg = fg^* = 0$ for any $f \in A$. Let $f = a_0x^k + a_1x^{k+1} + \cdots + a_sx^{k+s} \in A$, where k and s are nonnegative integers. Since R is $\alpha - * -$skew $* -$Armendariz, $\alpha^k(b_0), \alpha^{k+1}(b_1), \ldots, \alpha^{k+s}(b_s) \in r_*(B) = eRe$, and so $b_0, b_1, \ldots, b_m \in eRe$ since α is a $* -$utomorphism and $\alpha(e) = e$. Hence there exists $c_0, c_1, \ldots, c_m \in R$ such that $g = b_0c_0 + b_1c_1x + \_\_\_ + b_mc_mx^m = e(b_0 + b_1x + \_\_\_ + b_mx^m)e \in eR[x,\alpha]e$. Consequently $eR[x,\alpha]e = r_{*R[x,\alpha]}(A)$. Therefore $R[x,\alpha]$ is a *-Baer.
For sufficiency, we prove the result for $R[x,\alpha]$. Let $R[x,\alpha]$ be a *-Baer and D be asubset of $R[x,\alpha]$. Since $R[x,\alpha]$ is a *-Baer, then there exists a projection $e(x) = e \in R$, by **(Lemma 1)**, such that $r_{*R[x,\alpha]}(D) = \mathrm{e}R[x,\alpha]e$. Hence $r_{*R}(D) = \mathrm{e}Re$,
since $r_{*R}(D) \subseteq r\_R[x](D) = eR[x]e$.

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From **[2, Proposition 10]**, we have the followin**Corollary 5.** Let R be a domain $*$ $-$ring. Then R is α $-*$ $-$skew $*$ $-$Armendariz.

3. Extensions of α $-*$ $-$skew $*$ $-$Armendariz *-rings

In this section, we generalize the property of α $-*$ $-$skew *-Armendariz to some know extensions; namely the polynomial *-ring, the Laurent polynomial *-ring, the localization of R to S and from Ore *-ring to its classical Quotient.

It is an interesting question whether R is α $-*$ $-$skew $*$ $-$Armendariz if and only if $R[\boldsymbol{x}]$ is α $-*$ $-$skew $*$ $-$Armendariz for any $*$ $-$endomorphism of a $*$ $-$ring R. In this paper give a partial positive answer to this question. Our next proposition is also connected with this question.

Proposition 5. Let R be a reduced $*$ $-$ring and α be a $*$ $-$monomorphism of R. Then R is α $-*$ $-$skew $*$ $-$Armendariz if and only if $R[\boldsymbol{x}]$ is α $-*$ $-$skew $*$ $-$Armendariz.

Proof. Assume that R is α $-*$ $-$skew $*$ $-$Armendariz. Then R is α-$*$ $-$rigid by **Proposition 1** and the hypothesis. We claim that $R[\boldsymbol{x}]$ is also α-$*$ $-$rigid. In fact, for any $f(x) = a_0 + a_1x + \cdots + a_nx^n$ in $R[x]$, where $a_0, a_1 \cdots, a_n \in R$, if
$f(x)\alpha f(x) = f(x)\alpha f^*(x) = 0$, then
$(a_0 + a_1x + \cdots + a_nx^n)(\alpha(a_1) + \alpha(a_2)x + \cdots + \alpha(a_n)x^n) = 0$. Comparing the constant term we have $a_0\alpha(a_0) = 0$, so $a_0 = 0$, since R is α-$*$ $-$rigid. Now we have $(a_1x + \cdots + a_nx^n)(\alpha(a_1) + \alpha(a_2)x + \cdots + \alpha(a_n)x^n) = 0$. It gives that $a_1\alpha(a_1) = 0$, and so $a_1 = 0$.Continuing this process, at last we have $a_0 = a_1 = \cdots = a_n$. Hence $f(x) = 0$. By **(Corollary 1)**, $R[\boldsymbol{x}]$ is α $-*$ $-$skew $*$ $-$Armendariz.

Conversely, assume that $R[x]$ is α $-*$ $-$skew $*$ $-$Armendariz. Then R is α $-*$ $-$skew $*$ $-$Armendariz since R is a subring of $R[x]$.

In addition, we can prove R is $\alpha -*-$skew $*-$Armendariz if and only if $R[\boldsymbol{x}]$ is $\alpha -*-$skew $*-$Armendariz provided $\alpha^t = I_R$ for some positive integer t.

Theorem 2. Let α be a $*-$endomorphism of a $*-$ring R and $\alpha^t = I_R$ for some positive integer t. Then R is $\alpha -*-$skew $*-$Armendariz if and only if $R[x]$ is $\alpha -*-$skew $*-$Armendariz.

Proof. Let R be a $\alpha -*-$skew $*-$Armendariz *-ring and $p(y)q(y) = p(y)q^*(y) = 0$ with $p(y) = f_0 + f_1y + \cdots + f_my^m, q(y) = g_0 + g_1y + \cdots + g_ny^n \in R[x][y,\alpha]$ with $f_i = a_{i_0} + a_{i_1}x + \cdots + a_{w_i}x^{w_i}, g_i = b_{j_0} + b_{j_1}x + \cdots + b_{u_j}x^{u_j}$ for all $0 \leq i \leq m, 0 \leq j \leq n$, where $a_{i_0}, \cdots, a_{w_i}, b_{j_0}, \cdots, b_{u_j} \in R$. Take a positive integer t such that $t > max\left\{deg\,(f_i), deg\big(g_j\big)\right\}$ for any $0 \leq i \leq m, 0 \leq j \leq n$, where the degree is as polynomials in $R[x]$ and the degree of the zero polynomial is taken to be zero. Then $p(x^{kt}) = f_0 + f_1x^{kt} + \cdot\ \cdot\ \cdot + f_mx^{mkt}, q(x^{kt}) = g_0 + g_1x^{kt} + \cdot\ \cdot\ \cdot + g_nx^{nkt} \in R[x]$ and the set of coefficients of the f_i's (resp., g_j's) equals the set of coefficients of the $p(x^{kt})$ (resp., $q(x^{kt})$). Since $p(y)q(y) = p(y)q^*(y) = 0$ and x commutes with elements of R, $p(x^{kt})q(x^{kt}) = p(x^{kt})q^*(x^{kt}) = 0$. Since R is $\alpha -*-$skew $*-$Armendariz, $a_{l_i}\alpha^{l_i}\left(b_{s_j}\right) = 0$ for all $0 \leq i \leq m, 0 \leq j \leq n, 0 \leq l_i \leq w_i$, and $0 \leq s_j \leq u_j$. Thus $f_i\alpha^i\big(g_j\big) = 0$.
Conversely, assume that $R[x]$ is $\alpha -*-$skew $*-$Armendariz. Then R is
$\alpha -*-$skew $*-$Armendariz since R is a subring of $R[x]$.
Even though a ring R is $\alpha -*-$skew $*-$Armendariz if R is $\alpha -$rigid, the converse does not hold. Moreover, a $\alpha -*-$skew $*-$Armendariz ring $R[x]$ does not imply that R is $\alpha -$ rigid, i.e., $R[x,\alpha]$ is $*-$ reduced. In fact, for a $*-$ring R and α in **[2, Example 5]**, it can be easily checked that $R[y] = (\mathbb{Z}_2[x])[y]$ is also $\alpha -$ skew Armendariz and so $\alpha -*-$skew $*-$Armendariz, however R is not $\alpha -$ rigid.

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Proposition 6. A *-ring R is $\alpha -* -$skew $* -$Armendariz if and only if R_T is
$\breve{\alpha} -* -$skew $* -$Armendariz.

Proof. By **(Proposition 2)**, it suffices to prove the necessary condition.

Let R be a $\alpha -* -$skew $* -$Armendariz *-ring and $P(x)Q(x) = P(x)Q^*(x) = 0$ with $P(x) = \sum_{i=0}^{m} \beta_i x^i$, $Q(x) = \sum_{j=0}^{n} \gamma_j x^j \in R[x, \breve{\alpha}]_T$, where $\beta_i = u^{-1}a_i, \gamma_j = v^{-1}b_j$, and $a_i;\ b_j \in R, u;\ v \in T$.
Hence

$$P(x)Q(x) = (u^{-1}a_0 + u^{-`1}a_1x + \cdots + u^{-1}a_mx^m)(v^{-1}b_0 + v^{-1}b_1x + \cdots + v^{-1}b_nx^n)$$
$$= u^{-1}v^{-1}a_0\alpha^0(b_0) + u^{-1}v^{-1}\big(a_0\alpha^0(b_1) + a_1\alpha(b_0)\big)x$$
$$+ \cdots + u^{-1}v^{-1}(a_0\alpha^0(b_n) + \cdots + a_m\alpha^m(b_n)$$
$$= (vu)^{-1}(a_0\alpha^0(b_0) + (a_0\alpha^0(b_1) + a_1\alpha(b_0))x + \cdots$$
$$+ (a_0\alpha^0(b_n) + \cdots + a_m\alpha^m(b_0)x^m$$
$$= (vu)^{-1}p(x)q(x) = 0;$$

$$P(x)Q^*(x) = (u^{-1}a_0 + u^{-`1}a_1x + \cdots + u^{-1}a_mx^m)(v^{-1^*}b_0^* + v^{-1^*}b_1^*x + \cdots + v^{-1^*}b_n^*x^n)$$
$$= u^{-1}v^{-1^*}a_0\alpha^0(b_0)^* + u^{-1}v^{-1^*}(a_0\alpha^0(b_1)^* + a_1\alpha(b_0)^*)x$$
$$+ \cdots + u^{-1}v^{-1^*}(a_0\alpha^0(b_n)^* + \cdots + a_m\alpha^m(b_n)^*)$$
$$= (v^*u)^{-1}(a_0\alpha^0(b_0)^* + (a_0\alpha^0(b_1)^* + a_1\alpha(b_0)^*)x + \cdots$$
$$+ (a_0\alpha^0(b_n)^* + \cdots + a_m\alpha^m(b_0)^*x^m$$

$= (v^*u)^{-1}p(x)q^*(x) = 0;$ since T is contained in the center of R, so $p(x)q(x) = p(x)q^*(x) = 0$.
By hypothesis $a_i\alpha^j(b_j) = 0$ which implies$\beta_i\breve{\alpha}^j(\gamma_j) = (\mathrm{vu})^{-1}a_i$
R_T is $\breve{\alpha} -* -$skew $* -$Armendariz.

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