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Existence local solution of a hyperbolic-parabolic model of vasculogenesis

Jamila Abuajila Lagha[1], Aisha Almukhtar Lagha[2]

Department of Mathematics, Faculty of Science
University of Zawia, Libya
g.lagha@zu.edu.ly, a.lagha@zu.edu.ly

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Abstract

This paper is concerned with existence the local solution of three dimensional compressible chemotaxis system (Cauchy problem of a hyperbolic- parabolic model of vasculogenesis) with chemoattractant. We show the existence of local solutions by the energy method, and using induction, integral by parts, Cauchy–Schwarz inequality, Holder inequality, and Gronwall's inequality in proving steps of local solution existence.

Keywords: Chemotaxis system, Cauchy problem of a hyperbolic–parabolic model of vasculogenesis, energy method, Cauchy–Schwarz inequality, Holder inequality and Gronwall's inequality.

وجود حل محلي لنموذج القطع المكافئ–الزائدي لتكوين الأوعية الدموية

جميلة أبوعجيلة لاغا، عائشة المختار لاغا

قسم الرياضيات، كلية العلوم، جامعة الزاوية
a.lagha@zu.edu.ly, g.lagha@zu.edu.ly

الملخص

يهتم هذا البحث بوجود الحل الموضعي لنظام الانجذاب الكيميائي ثلاثي الأبعاد (مسألة كوشي لنموذج القطع الزائد والقطع المكافئ لتكوين الأوعية الدموية) مع الجاذب الكيميائي. وقد بينا وجود حلول محلية بطريقة الطاقة؛ وأيضا باستخدام الاستنتاج

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الرياضي؛ التكامل بالأجزاء؛ متباينة هولدر؛ متباينة كوشي– شوارتز ومتباينة غرونوال في إثبات خطوات وجود الحل الموضعي.

الكلمات الدالة: نظام الانجذاب الكيميائي، مشكلة كوشي لنموذج القطع مكافئ– زائدي لتكوين الأوعية الدموية، طريقة الطاقة، متباينة كوشي – شوارتز، متباينة هولدر و متباينة جرونوال.

1. Introduction

Chemotaxis is a biological process describe the movement of an organism orentity in response to a chemical stimulus. Somatic cells, bacteria, and other single-cell or multicellular organisms direct their movements according to certain chemicals in their environment, and models for chemotaxis have been successfully applied to the aggregation patterns in bacteria [1], [2], slime molds, skin pigmentation patterns [3], angiogenesis in tumor progression and wound healing [4], and many other examples. Therefore, a huge number of works, both theoretical and experimental, have been dedicated to exploring and hence understanding the mechanistic basis of chemotaxis.

In 1953, Patlak [5] contributed the first mathematical idea to model chemotaxis. In 1970s, Keller and Segel [6], [7] proposed a classical and fundamental mathematical system to model chemotaxis. These pioneering works have initiated an intensive mathematical investigation of the (Patlak et al) model over the last 40 years. They initiated a fruitful and have become one of the best-studied models in mathematical biology and still continuing period of mathematical analysis of chemotaxis by introducing a system of partial differential equations, the general form of which

$$\begin{cases} n_t = \nabla n - \nabla.(n\chi\nabla\phi), \\ \alpha\phi_t = D\Delta\phi + g(\phi, n). \end{cases} \qquad \text{(A)}$$

Where $n(x,t)$is the cell density, $\phi(x,t)$ is the concentration of chemical attractant, χ is the sensitivity of the cell movement to the

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density gradient of the attractant, α is a positive constant, and the reaction term g is a smooth function of the arguments .

There are two limiting cases of the Keller-Segel model (A). The first one is when the chemical substance relaxes so fast that it reaches its equilibrium instantaneously, i.e., $\alpha \rightarrow 0$ and (A) is reduced to a parabolic-elliptic system, see Ref [8]. The second one is when the diffusion of the chemical substance is so small that it is negligible, i.e, $D \longrightarrow 0$.

Hattori and Lagha [9, 10] studied Global Existence and Decay Rates of the Solutions for the compressible Chemotaxis System with chemoattractant and repellent in three dimensions with Lotka-Volterra type model for Chemo Agents, they used the Fourier transform and energy method to accomplish that.

In this paper we consider the following Cauchy problem of a hyperbolic-parabolic model of vasculogenesi

$$\begin{cases} \partial_t \rho + div(\rho u) = 0 \\ \partial_t (\rho u) + div(\rho u \otimes u) + \nabla \mathrm{P}(\rho) = \mu\rho\nabla\phi - \lambda\rho u \\ \partial_t \phi = \Delta\phi + a\rho\phi - b\phi \qquad t > 0\,,\;\; x \in R^3. \end{cases} \qquad (B)$$

Where $\rho(x,t)$ is the density of endothelial cells $u(x,t)$, is the cell velocity, and $\phi(x,t)$ is the concentration of endothelial cells chemotactic factor, P is a monotonic pressure function. The positive constants a and b denote the secretion and death rates of the chemoattractant, respectively. In addition, the other two positive constants μ and λ respectively measure the friction force of the cell on the substrate and the reaction strength of the cells to chemical signals.

The parabolic-hyperbolic coupled system of partial differential equations may arise from physics, mechanics and material science such as the compressible Navier-Stokes equations, thermo (visco) elastic systems. The properties of solutions to nonlinear parabolic-hyperbolic coupled systems are very different from those of parabolic or hyperbolic equations. There are many mathematical researches for various parabolic-hyperbolic coupled systems on the well-posedness (local and global) and asymptotical behavior of

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solutions since 1970s (cf. [11]. It is well-known that the diffusion (parabolic) dissipation can smoothen solutions from the crude initial data, while on the contrary the hyperbolic effect can coarsen solutions from the smooth initial data (e.g. see [12]). Therefore, the relation between the regularity of solutions and the initial values of parabolic-hyperbolic systems has been an interesting topic and attracted many studies (e.g. see [13], [14]).

When $\phi = 0$ (or without the effect of chemotaxis), the system (B) becomes the well-known Euler equations with damping. The global well-posedness and L^2-decay rate of small solution of the three dimensional Euler equations with damping in Sobolev space were considered in [15, 16, 17]. In [18, 19] focus are considered on mathematical models of in vitro vasculogenesis are considered where it showed experiments of in vitro formation of blood vessels show that cells randomly spread on a gel matrix autonomously organize to form a connected vascular network.

Later, Fang and Xu [20] and Jiu and Zheng [21] the existence and asymptotic behavior of solutions of the multi-dimensional compressible Euler equations with damping on the framework of Besov space are considered. Xu and Wang [22] derived relaxation limit in Besov spaces. Liao et al [23] studied L^P-convergence rates of planar waves, and Deng [24] gave the pointwise description of the solution for the half space problem more precisely, we also refer to [25, 26, 27, 28, and 29] for the so-called p-system with damping and Huang et. Mengqian Liu and Zhigang Wu [30] studied the Cauchy problem of a hyperbolic- parabolic model of vasculogenesis in dimension three and they got it the optimal L^2-decay rate of the solution and its highest order derivatives when the initial perturbation is small in $H^N(R^3)$ and bounded in $L^1(R^3)$.

The main goal of this paper is to establish the local and existence of smooth solution of Cauchy problem of a hyperbolic- parabolic model of vasculogenesis in three dimensions around a constant state $(\rho_\infty, 0,0)$.

The main result of this paper is stated as follows.

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Theorem 1.1.
Let $N \geq 0$ be an integer, there are constants $0 \leq T < \infty$ and $\varepsilon_0 > 0$ such that if the initial data $W_0 \in H^N(R^3)$ and $\|W_0\| \leq \varepsilon_0$ then there exists a unique solution $W = (\rho, u, \phi)$ of the Cauchy problem (2.1)–(2.2) on $[0, T]$ with

$$(\rho - \rho_\infty, u) \in C([0,T]; H^N(R^3)) \cap C^1([0,T]; H^{N-1}(R^3),$$

$$\phi \in C\left([0,T]; H^N(R^3) \cap C^1(0,T); H^{N-2}(R^3)\right).$$

We will also consider the simplified Chemotaxis fluid equations in the form of the following initial value problem

$$\begin{cases} \partial_t \rho + \nabla.(\rho u) = 0, \\ \partial_t u + u.\nabla \mathrm{u} + \frac{\nabla \mathrm{P}(\rho)}{\rho} = \mu \nabla \phi - \lambda u, \\ \partial_t \phi - \Delta \phi = a\rho\phi - b\phi, \end{cases} \qquad (1.1)$$

With initial data:

$$(\rho, u, \phi)|_{t=0} = (\rho_0(x), u_0(x), \phi_0(x)). \qquad (1.2)$$

Where it is supposed to hold that $(\rho_0(x), u_0(x), \phi_0(x)) \to (\rho_\infty, 0, 0)$, as $|x| \longrightarrow 0$, for some constant $\rho_\infty > 0$, where In what follows, the integer $N \geq 4$ is always assumed. We give some notations, C denotes some positive constant and γ_i where i = 1, 2, denotes some positive (generally small) constant, where both C and γ_i which may vary in different estimates. For any integer $N \geq 0$ and $P \geq 1$, the Sobolev space $W^{N,P}(R^3)$ denoted as $H^N(R^3)$ when, $P = 2$, also when $N = 0$ and $P = 2$ which becomes $L^2 = H^0$. We set $\partial^k = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \partial_{x_3}^{k_3}$ for a multi-index $k = \{k_1, k_2, k_3\}$, the length of k is $|k| = \sum_{i=1}^{3} k_i$.
This paper is organized as follows. In Section 2, we reformulate the Cauchy problem under consideration. In Section 3, we prove the local existence and uniqueness of solutions.

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2. Reformulation of the system.

Let $W(t) = (\rho, u, \phi)$ be a smooth solution to the Cauchy problem of the chemotaxis system (1.1) with initial data $W_0 = (\rho_0, u_0, \phi_0)$.

Let us adjust the variables $\rho = \rho_\infty + n(x,t)$, so that the Cauchy problem (1.1) is rewritten as follows for ease of presentation in the future:

$$\begin{cases} \partial_t n + \rho_\infty \nabla u + \nabla.(nu) = 0, \\ \partial_t u + u.\nabla u + \frac{\nabla P'(n+\rho_\infty)}{n+\rho_\infty} = \mu \nabla \phi - \lambda u, \\ \partial_t \phi - \Delta \phi + (b - a\rho_\infty)\phi = an\phi, \end{cases} \quad (2.1)$$

With initial data**:**

$$(n, u, \phi)|_{t=0} = (n_0, u_0, \phi_0) \to (0,0,0). \quad (2,2)$$

As $|x| \to 0$, where $n_0 = \rho_0 - \rho_\infty$. We assume that $b - a\rho_\infty > 0$.

3. Existence of local solutions.

In this paper, we show the proof of the existence of local solutions (n, u, ϕ) by constructing sequence of functions that converges to a function satisfying the Cauchy problem. We construct a solution sequence $(n^j, u^j, \phi^j)_{j \geq 0}$, by iteratively solving the Cauchy problem on the following:

$$\begin{cases} \partial_t n^{j+1} + (n^j + \rho_\infty)\nabla.u^{j+1} + \nabla n^{j+1} u^j = 0, \\ \partial_t u^{j+1} + u^j.\nabla u^{j+1} + \frac{P'(n^j+\rho_\infty)}{n^j+\rho_\infty}\nabla n^{j+1} = \mu \nabla \phi^{j+1} - \lambda u^{j+1}, \\ \partial_t \phi^{j+1} - \Delta \phi^{j+1} + (b - a\rho_\infty)\phi^{j+1} = an^j \phi^{j+1}. \end{cases} \quad (3.1)$$

With initial data:

$$(n^j, u^j, \phi^j)_{t=0} = W_0 = (n_0, u_0, \phi_0) \quad (3.2)$$

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For simplicity, in what follows, we write $W^j = (n^j, u^j, \phi^j)$ and $W_0 = (n_0\,, u_0\,, \phi_0)$ where $W^{\,0} = (0{,}0{,}0)$.

Lemma 3.1.
There are constants $\varepsilon_0 > 0$, $T_1 > 0$, $M > 0$ such that if $\|W_0\|_{H^N} \le \varepsilon_0$, then for each $j \ge 0$, $W^j \in C\big([0, T_1]: H^N(R^3)\big)$ is well defined and

$$\sup_{0 \le t \le T_1} \left\|W^j(t)\right\|_{H^N} \le M, \quad j \ge 0. \qquad \text{(C)}$$

Moreover, $(W^j)_{j\ge0}$ is a Cauchy sequence in the Banach space $C([0, T_1]; H^N(R^3)))$, and the limit function $W(x,t)$ of $(W^j)_{j\ge0}$ satisfies:

$$\sup_{0 \le t \le T_1} \|W(t)\|_{H^N} \le M. \qquad \text{(D)}$$

With $W = (n, u, \phi)$ is a solution given by the Cauchy problem (3.1)-(3.2) over $[0, T_1]$. At last, the Cauchy problem (3.1)-(3.2) admits at most one solution $W \in C\big([0, T_1]: H^N(R^3)\big)$, which satisfies (D).

Proof:
We set $W^{\,0} = (0{,}0{,}0)$. Then, we use W^0 to solve the equations for W^1. The first and the second equation are the first order partial differential equation and the third equation is the second order parabolic equations. We obtain $n^1(x,t)$, $u^1(x,t)$ and $\phi^1(x,t)$ in this order. Similarly, we define $\big(n^j, u^j, \phi^j\big)$ iteratively. Now, we prove the existence and uniqueness of solutions in space $C\big([0, T_1]: H^N(R^3)\big)$, where $T_1 > 0$, is suitably small. Let us divided the proof into four steps as follows:
Step one, in this step, via energy estimates we will show the uniform Boundedness of the sequence of functions under our construction. We show that there exists a constant $M > 0$ such that $W^j \in C\big([0, T_1]: H^N(R^3)\big)$ is well defined and

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$$\sup_{0\le t\le T_1} \left\|W^j\right\|_{H^N} \le M, \quad \text{for all } j \ge 0 . \qquad (3.3)$$

We can prove (3.3) by using the induction.

It is trivial when $j = 0$. Suppose that it is true for $j \ge 0$, where M is small enough. To prove for $j+1$, we need some energy estimate for W^{j+1}. Now, we make estimates on the high-order derivatives of (n, u, ϕ). Take k with $|k| \le N$.
We can Apply ∂^k to the second equation of (3.1), multiplying by $\partial^k u^{j+1}$ and then integrating in **x**, we have

$$\int_{R^3} \partial^k u^{j+1} \partial^k \partial_t u^{j+1} dx + \int_{R^3} \partial^k u^{j+1} \partial^k (u^j . \nabla \mathrm{u}^{j+1}) dx$$
$$+ \int_{R^3} \partial^k u^{j+1} \partial^k \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \nabla \mathrm{n}^{j+1} dx$$
$$= \mu \int_{R^3} \partial^k u^{j+1} \partial^k \nabla \phi^{j+1} dx - \lambda \int_{R^3} \partial^k u^{j+1} \partial^k u^{j+1} dx.$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_{R^3} \left(\partial^k u^{j+1}\right)^2 dx + \lambda \int_{R^3} \left|\partial^k u^{j+1}\right|^2 dx$$
$$+ \int_{R^3} \partial^k u^{j+1} \left(\frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty}\right) \partial^k \nabla n^{j+1} dx$$
$$= - \int_{R^3} \partial^k u^{j+1} \sum_{i=1}^{k} C_i^k \, \partial^i \left(\frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty}\right) \partial^{k-i} \nabla n^{j+1} dx$$
$$- \int_{R^3} \partial^k u^{j+1} \sum_{i=0}^{k} C_i^k \left(\partial^i u^j \nabla \partial^{k-i} u^{j+1}\right) dx$$
$$+ \mu \int_{R^3} \left|\partial^k u^{j+1}\right| \left|\partial^k \nabla \phi^{j+1}\right| dx. \qquad (3.4)$$

Where the right-hand side is bounded by

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$$C\|n^j\|_{H^N}\|u^{j+1}\|_{H^N}\|n^{j+1}\|_{H^N}+C\|u^j\|_{H^N}\|u^{j+1}\|_{H^N}^2$$
$$+C\|u^{j+1}\|_{H^N}\|\nabla\phi^{+1}\|_{H^N}.$$

We estimate on the third term from the left-hand side of the previous equality and by using integrating by parts, we have

$$\int_{\boldsymbol{R}^3}\partial^k u^{j+1}\left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\right)\partial^k\nabla n^{j+1}dx$$
$$=-\int_{\boldsymbol{R}^3}\partial^k\nabla u^{j+1}\left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\right)\partial^k n^{j+1}dx$$
$$-\int_{\boldsymbol{R}^3}\partial^k u^{j+1}\nabla\left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\right)\partial^k n^{j+1}d\,x.$$

By compensation $\left\{\nabla u^{j+1}=-\left(\frac{1}{n^j+\rho_\infty}\partial_t n^{j+1}+\frac{\nabla n^{j+1}.u^j}{n^j+\rho_\infty}\right)\right\}$ from the first equation (1.3) and by using integration by parts, one has

$$\int_{\boldsymbol{R}^3}\partial^k u^{j+1}\left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\right)\partial^k\nabla n^{j+1}$$
$$=\int_{\boldsymbol{R}^3}\partial^k\left(\frac{1}{n^j+\rho_\infty}\partial_t n^{j+1}\right.$$
$$\left.+\frac{\nabla n^{j+1}.u^j}{n^j+\rho_\infty}\right)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx$$

$$-\int_{\boldsymbol{R}^3}\partial^k u^{j+1}\nabla\left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\right)\partial^k n^{j+1}dx$$
$$=\frac{1}{2}\int_{\boldsymbol{R}^3}\frac{\mathrm{P}'(n^j+\rho_\infty)}{(n^j+\rho_\infty)^2}(\partial^k n^{j+1})_t^2dx$$
$$+\int_{\boldsymbol{R}^3}\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k\left(\frac{1}{n^j+\rho_\infty}\right)\partial_t n^{j+1}\partial^k n^{j+1}dx$$
$$+\int_{\boldsymbol{R}^3}\left(\partial^k\frac{1}{n^j+\rho_\infty}\right)(\nabla n^{j+1}.u^j)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx$$

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$$+\int_{R^3}\frac{1}{n^j+\rho_\infty}\partial^k(\nabla n^{j+1}.u^j)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx$$
$$-\int_{R^3}(\partial^k u^{j+1})\frac{\nabla\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx. \qquad (3.5)$$

With compensation from (3.5) in (3.4), we get

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}\left(\partial^k u^{j+1}\right)^2dx+\lambda\int_{R^3}\left|\partial^k u^{j+1}\right|^2dx$$
$$+\frac{1}{2}\int_{R^3}\frac{\mathrm{P}'(n^j+\rho_\infty)}{(n^j+\rho_\infty)^2}(\partial^k n^{j+1})_t^2dx$$
$$=-\int_{R^3}\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k\left(\frac{1}{n^j+\rho_\infty}\right)\partial_t n^{j+1}\partial^k n^{j+1}dx$$
$$-\int_{R^3}\left(\partial^k\frac{1}{n^j+\rho_\infty}\right)(\nabla n^{j+1}.u^j)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx$$
$$-\int_{R^3}\frac{1}{n^j+\rho_\infty}\partial^k(\nabla n^{j+1}.u^j)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx$$
$$+\int_{R^3}(\partial^k u^{j+1})\frac{\nabla\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx$$
$$-\int_{R^3}\partial^k u^{j+1}\sum_{i=1}^{k}C_i^k\,\partial^i\left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\right)\partial^{k-i}\nabla n^{j+1}dx$$
$$-\int_{R^3}\partial^k u^{j+1}\sum_{i=0}^{k}C_i^k\,(\partial^i u^j\nabla\partial^{k-i}u^{j+1})dx$$
$$+\mu\int_{R^3}\left|\partial^k u^{j+1}\right|\left|\partial^k\nabla\phi^{j+1}\right|dx. \qquad (3.6)$$

Where these terms

$$-\int_{R^3}\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k\left(\frac{1}{n^j+\rho_\infty}\right)\partial_t n^{j+1}\partial^k n^{j+1}dx$$

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$$
\begin{aligned}
&-\int_{R^3}\left(\partial^k\left(\frac{1}{n^j+\rho_\infty}\right)\right)(\nabla n^{j+1}u^j)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx\\
&-\int_{R^3}\frac{1}{n^j+\rho_\infty}\partial^k(\nabla n^{j+1}.u^j)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx\\
&+\int_{R^3}\partial^k u^{j+1}\frac{\nabla\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k n^{j+1}dx,
\end{aligned}
$$

are bounded by

$$
\begin{aligned}
&C\|n^j\|_{H^N}\|n^{j+1}\|_{H^N}+C\|u^j\|_{H^N}\|n^{j+1}\|_{H^N}+C\|u^j\|_{H^N}\|n^{j+1}\|_{H^N}^2\\
&+C\|n^j\|_{H^N}\|n^{j+1}\|_{H^N}\|u^{j+1}\|_{H^N}. \qquad (3.7)
\end{aligned}
$$

With compensation (3.5)-(3.7) in (3.4), after taking summation over $|k|\le N$ and by using the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^{j+1}\|_{H^N}^2+\lambda\|u^{j+1}\|_{H^N}^2+\frac{1}{2}\frac{d}{dt}\|n^{j+1}\|_{H^N}^2\le C\|n^j\|_{H^N}^2\\
&+C\|n^{j+1}\|_{H^N}^2+C\|n^j\|_{H^N}^2\|n^{j+1}\|_{H^N}^2+C\|u^{j+1}\|_{H^N}^2\\
&+C\|u^j\|_{H^N}^2\|u^{j+1}\|_{H^N}^2+C\|u^j\|_{H^N}^2+C\|\nabla\phi^{j+1}\|_{H^N}^2\\
&+C\|u^j\|_{H^N}^2\|n^{j+1}\|_{H^N}^2+C\|n^j\|_{H^N}^2\|u^{j+1}\|_{H^N}^2 .
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|n^{j+1}\|_{H^N}^2+\|u^{j+1}\|_{H^N}^2\right)+\gamma_2\|u^{j+1}\|_{H^N}^2\\
&\le C\|n^j\|_{H^N}^2+C\|u^j\|_{H^N}^2+C\|n^{j+1}\|_{H^N}^2\\
&+C\|n^j\|_{H^N}^2\|n^{j+1}\|_{H^N}^2+C\|u^j\|_{H^N}^2\|u^{j+1}\|_{H^N}^2\\
&+C\|u^j\|_{H^N}^2\|n^{j+1}\|_{H^N}^2+C\|n^j\|_{H^N}^2\|u^{j+1}\|_{H^N}^2\\
&+C\|\nabla\phi^{j+1}\|_{H^N}^2. \qquad (3.8)
\end{aligned}
$$

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By the same method as above, we apply ∂^k to the third equation of (3.1), multiplying by $\partial^k \phi^{j+1}$ and integrating in x, we obtain

$$\int_{\boldsymbol{R}^3} \partial^k \phi^{j+1} \partial^k \, \partial_t \phi^{j+1} dx - \int_{\boldsymbol{R}^3} \partial^k \phi^{j+1} \partial^k \Delta \phi^{j+1} dx$$
$$+(b - a\rho_\infty) \int_{\boldsymbol{R}^3} \left(\partial^k \phi^{j+1}\right)^2 dx \; = a \int_{\boldsymbol{R}^3} \partial^k \phi^{j+1} \partial^k (n^j \, \phi^{j+1}) dx.$$

By using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{R}^3} \left(\partial^k \phi^{j+1}\right)^2 dx + \int_{\boldsymbol{R}^3} \partial^k \nabla \phi^{j+1} \partial^k \nabla \phi^{j+1} dx$$
$$+(b - a\rho_\infty) \int_{\boldsymbol{R}^3} (\partial^k \phi^{j+1})^2 dx$$
$$= a \int_{\boldsymbol{R}^3} \partial^k \phi^{j+1} \sum_{i=0}^{k} C_i^k \, (\partial^i n^j \partial^{k-i} \phi^{j+1}) dx.$$

Applying Holder inequality, we get

$$\frac{1}{2}\frac{d}{dt}\left\|\phi^{j+1}\right\|_{H^N}^2 + \left\|\nabla \phi^{j+1}\right\|_{H^N}^2 + C\left\|\phi^{j+1}\right\|_{H^N}^2$$
$$\leq C\left\|n^j\right\|_{H^N}\left\|\phi^{j+1}\right\|_{H^N}^2.$$

By using the Cauchy-Schwarz inequality, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\phi^{j+1}\right\|_{H^N}^2 + \left\|\nabla \phi^{j+1}\right\|_{H^N}^2 + \gamma_2\left\|\phi^{j+1}\right\|_{H^N}^2$$
$$\leq C\left\|n^j\right\|_{H^N}^2\left\|\phi^{j+1}\right\|_{H^N}^2, +C\left\|n^j\right\|_{H^N}^2. \qquad (3.9)$$

Then, by taking summation of (3.8) ,(3.9), we have

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$$\frac{d}{dt}\left(C_1\|\phi^{j+1}\|_{H^N}^2+\|\mathrm{u}^{j+1}\|_{H^N}^2+\|\phi^{j+1}\|_{H^N}^2\right)+\gamma_1\|\nabla\phi^{j+1}\|_{H^N}^2$$
$$+\gamma_2(\|\mathrm{u}^{j+1}\|_{H^N}^2+\|\phi^{j+1}\|_{H^N}^2)\le C(\|n^j\|_{H^N}^2+\|u^j\|_{H^N}^2)$$
$$+C\|n^{j+1}\|_{H^N}^2+C\|n^j\|_{H^N}^2\|n^{j+1}\|_{H^N}^2+C\|u^j\|_{H^N}^2\|u^{j+1}\|_{H^N}^2$$
$$+C\|n^j\|_{H^N}^2\|u^{j+1}\|_{H^N}^2+C\|u^j\|_{H^N}^2\|n^{j+1}\|_{H^N}^2$$
$$+C\|n^j\|_{H^N}^2\|\phi^{j+1}\|_{H^N}^2.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\left(C_1\|\phi^{j+1}\|_{H^N}^2+\|\mathrm{u}^{j+1}\|_{H^N}^2+\|\phi^{j+1}\|_{H^N}^2\right)$$
$$+\gamma_1\|\nabla\phi^{j+1}\|_{H^N}^2+\gamma_2(\|\mathrm{u}^{j+1}\|_{H^N}^2+\|\phi^{j+1}\|_{H^N}^2)$$
$$\le C(\|n^j\|_{H^N}^2+\|u^j\|_{H^N}^2)+C\|u^j\|_{H^N}^2\left(\|n^{j+1}\|_{H^N}^2+\|\mathrm{u}^{j+1}\|_{H^N}^2\right)$$
$$+C\|u^j\|_{H^N}^2\left(\|n^{j+1}\|_{H^N}^2+\|\mathrm{u}^{j+1}\|_{H^N}^2\right)$$
$$+C\|n^j\|_{H^N}^2\left(\|n^{j+1}\|_{H^N}^2+\|\mathrm{u}^{j+1}\|_{H^N}^2+\|\phi^{j+1}\|_{H^N}^2\right)$$
$$+C\|n^{j+1}\|_{H^N}^2. \qquad (3.10)$$

Then, after integrating with respect to t, we have

$$\|W^{j+1}\|_{H^N}^2+\gamma_1\int_0^t\|\nabla\phi^{j+1}\|_{H^N}^2ds+\gamma_2\int_0^t\|(\mathrm{u}^{j+1},\phi^{j+1})\|_{H^N}^2ds$$
$$\le\|W^{j+1}(0)\|_{H^N}^2+C\int_0^t\|W^j\|_{H^N}^2ds$$
$$+C\int_0^t\|W^j\|_{H^N}^2\|(\mathrm{n}^{j+1},u^{j+1})\|_{H^N}^2ds$$
$$+C\int_0^t\|W^j\|_{H^N}^2\|(\mathrm{n}^{j+1},u^{j+1},\phi^{j+1})\|_{H^N}^2ds$$
$$+C\int_0^t\|n^{j+1}\|_{H^N}^2ds.$$

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Thus

$$\left\|W^{j+1}\right\|_{H^N}^2+\gamma_1\int_0^t\left\|\nabla\phi^{j+1}\right\|_{H^N}^2 ds+\gamma_2\int_0^t\left\|(\mathrm{u}^{j+1},\phi^{j+1})\right\|_{H^N}^2 ds$$
$$\leq\left\|W^{j+1}(0)\right\|_{H^N}^2+C\int_0^t\left\|n^{j+1}\right\|_{H^N}^2 ds$$
$$+C\int_0^t\left\|W^j\right\|_{H^N}^2 ds+C\int_0^t\left\|W^j\right\|_{H^N}^2\left\|(\mathrm{n}^{j+1},u^{j+1},\phi^{j+1})\right\|_{H^N}^2 ds.$$

By using Gronwall's inequality, we obtain

$$\sup_{0\leq t\leq T_1}\left\|W^{j+1}(t)\right\|_{H^N}^2\leq e^{\int_0^t C ds}\int_0^t\left\|W^j(s)\right\|_{H^N}^2 ds$$
$$+e^{\int_0^t C ds}\left\|W^{j+1}(0)\right\|_{H^N}^2\leq C\varepsilon_0^2 T_1 e^{CT_1}\sup_{0\leq t\leq T_1}\left\|W^j\right\|_{H^N}^2 .$$

From the previous inductive assumption, we get

$$\left\|W^{j+1}\right\|_{H^N}^2+\gamma_1\int_0^t\left\|\nabla\phi^{j+1}\right\|_{H^N}^2 ds+\gamma_2\int_0^t\left\|(\mathrm{u}^{j+1},\phi^{j+1})\right\|_{H^N}^2 ds\leq C\varepsilon_0 T_1 M^2. \quad (3.11)$$

We now choose suitable small constants $\varepsilon_0>0, T_1>0$ and $M>0$ such that:

$$\left\|W^{j+1}\right\|_{H^N}^2+\gamma_1\int_0^t\left\|\nabla\phi^{j+1}\right\|_{H^N}^2 ds+\gamma_2\int_0^t\left\|(\mathrm{u}^{j+1},\phi^{j+1})\right\|_{H^N}^2 ds\leq M^2. \quad (3.12)$$

For any $0\leq t\leq T_1$. This implies that (3.3) holds true for $j+1$ if so for j. Hence (3.3) is proved.

Step 2, we prove that the sequence $(W^J)_{j\geq 0}$ is a Cauchy sequence in the Banach space $C\big([0,T_1]:H^N(R^3)\big)$ which converges to the solution $W=(n,u,\phi)$ of the Cauchy problem

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(3.1)-(3.2), and satisfies $\sup_{0\le t\le T_1}\|W^j\|_{H^N}\le M$. For simplicity, we denote $\delta f^{j+1}=f^{j+1}-f^j$ Subtracting the j-th equations from the $j+1$-th equations, we have the following equations for $\delta n^{j+1},\delta u^{j+1}$and $\delta\phi^{j+1}$

$$\begin{cases}\partial_t\delta n^{j+1}+(n^j+\rho_\infty)\nabla\delta u^{j+1}\\ =(\delta n^j+\rho_\infty)\nabla u^j-\delta u^j.\nabla n^j-u^j\nabla\delta n^{j+1},\\ \partial_t\delta u^{j+1}+\delta u^j.\nabla\mathrm{u}^j+u^j\nabla\delta\mathrm{u}^{j+1}+\lambda\delta u^{j+1}\\ =\mu\nabla\delta\phi^{j+1}+\delta n^j.\nabla\mathrm{n}^j-\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\delta\nabla\mathrm{n}^{j+1} \quad ,\\ \partial_t\delta\phi^{j+1}-\Delta\delta\phi^{j+1}+(b-a\rho_\infty)\delta\phi^{j+1}\\ =an^j\delta\phi^{j+1}+a\phi^j\delta n^j\,.\end{cases}\tag{3.13}$$

By using the same estimates as before for second equation, we have

$$\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k\partial_t\delta u^{j+1}dx+\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k(\delta u^j.\nabla\mathrm{u}^j)dx$$

$$+\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k(u^j.\nabla\delta\mathrm{u}^{j+1})dx+\lambda\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k\delta u^{j+1}dx$$
$$=\mu\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k\delta\nabla\phi^{j+1}dx-\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k(\delta n^j.\nabla\mathrm{n}^j)dx$$
$$-\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\nabla\delta\mathrm{n}^{j+1}dx.$$

Thus

$$\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k\partial_t\delta u^{j+1}dx$$
$$+\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\sum_{i=0}^{k}C_i^k\,\partial^i\delta u^j\partial^{k-i}\nabla\mathrm{u}^j)dx$$

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$$
\begin{aligned}
&+\int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \sum_{i=0}^{k} C_i^k \,(\partial^i u^j \partial^{k-i} \nabla \delta u^{j+1}) dx \\
&+\lambda \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \partial^k \delta u^{j+1}) dx = \mu \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \partial^k \nabla \delta \phi^{j+1}) dx \\
&- \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \sum_{i=0}^{k} C_i^k \,\partial^i \delta n^j \partial^{k-i} \nabla \mathrm{n}^j) dx \\
&- \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \partial^k \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \nabla \delta \mathrm{n}^{j+1} dx \\
&- \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \sum_{i=1}^{k} C_i^k \,\partial^i \left(\frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \right) \partial^{k-i} \nabla \delta n^{j+1} dx \,. \quad (3.14)
\end{aligned}
$$

We can estimate the third term on the right-hand side of the previous equality by using integration by parts and using the first equation of (3.13), we get

$$
\begin{aligned}
&\int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \partial^k \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \nabla \delta n^{j+1} dx \\
&= -\int_{\boldsymbol{R}^3} \partial^k \nabla \delta u^{j+1} \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx \\
&\quad - \int_{R^3} \partial^k \delta u^{j+1} \,\nabla \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx. \\
&= \int_{\boldsymbol{R}^3} \partial^k \left(\frac{1}{n^j + \rho_\infty} \partial_t \delta n^{j+1} + \frac{(\delta n^j + \rho_\infty) \nabla u^j}{n^j + \rho_\infty} + \frac{\nabla n^j \delta u^j}{n^j + \rho_\infty} \right. \\
&\qquad \left. + \frac{u^j \nabla \delta n^{j+1}}{n^j + \rho_\infty} \right) \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx \\
&\quad - \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \,\nabla \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx.
\end{aligned}
$$

Thus

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$$
\begin{aligned}
&= \int_{\boldsymbol{R}^3} \partial^k \left(\frac{1}{n^j + \rho_\infty} \partial_t \delta n^{j+1} \right. \\
&\qquad\qquad \left. + \frac{(\delta n^j + \rho_\infty) \nabla u^j}{n^j + \rho_\infty} \right) \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx \\
&+ \int_{\boldsymbol{R}^3} \partial^k \left(\frac{\nabla n^j \delta u^j}{n^j + \rho_\infty} + \frac{u^j \nabla \delta n^{j+1}}{n^j + \rho_\infty} \right) \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx \\
&- \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \nabla \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx \\
&= \frac{1}{2} \int_{\boldsymbol{R}^3} \frac{\mathrm{P}'(n^j + \rho_\infty)}{(n^j + \rho_\infty)^2} (\partial^k \, \delta n^{j+1})_t^2 \\
&+ \int_{\boldsymbol{R}^3} \frac{\mathrm{P}'(n^j + \rho_\infty)}{(n^j + \rho_\infty)} \partial^k \left(\frac{1}{n^j + \rho_\infty} \right) \partial_t \delta n^{j+1} \partial^k \delta n^{j+1} dx \\
&+ \int_{\boldsymbol{R}^3} \frac{1}{n^j + \rho_\infty} \partial^k ((\delta n^j + \rho_\infty) \nabla u^j) \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx \\
&+ \int_{\boldsymbol{R}^3} \left(\partial^k \left(\frac{1}{n^j + \rho_\infty} \right) \right) ((\delta n^j \\
&\qquad\qquad + \rho_\infty) \nabla u^j) \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx \\
&+ \int_{\boldsymbol{R}^3} \partial^k \left(\frac{\nabla n^j \delta u^j}{n^j + \rho_\infty} + \frac{u^j \nabla \delta n^{j+1}}{n^j + \rho_\infty} \right) \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx \\
&- \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1} \nabla \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx. \qquad (3.15)
\end{aligned}
$$

Then these terms

$$
\begin{aligned}
&- \int_{\boldsymbol{R}^3} \frac{\mathrm{P}'(n^j + \rho_\infty)}{(n^j + \rho_\infty)} \partial^k \left(\frac{1}{n^j + \rho_\infty} \right) \partial_t \delta n^{j+1} \partial^k \delta n^{j+1} dx \\
&- \int_{\boldsymbol{R}^3} \frac{1}{n^j + \rho_\infty} \partial^k ((\delta n^j + \rho_\infty) \nabla u^j) \frac{\mathrm{P}'(n^j + \rho_\infty)}{n^j + \rho_\infty} \partial^k \delta n^{j+1} dx
\end{aligned}
$$

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$$
\begin{aligned}
&-\int_{\boldsymbol{R}^3}\left(\partial^k\left(\frac{1}{n^j+\rho_\infty}\right)\right)(\,(\delta n^j\\
&\qquad\qquad+\rho_\infty)\nabla u^j)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k\delta n^{j+1}dx\\
&+\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\nabla\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k\delta n^{j+1}dx\\
&-\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\sum_{i=1}^{k}C_i^k\,\partial^i\left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\right)\partial^{k-i}\nabla\delta n^{j+1}dx,
\end{aligned}
$$

are bounded by

$$
\begin{aligned}
&C\left\|\delta u^j\right\|_{H^{N-1}}\left\|\delta n^{j+1}\right\|_{H^{N-1}}+C\left\|\delta n^j\right\|_{H^{N-1}}\left\|\delta n^{j+1}\right\|_{H^{N-1}}\\
&+C\left\|n^j\right\|_{H^{N-1}}\left\|\delta u^{j+1}\right\|_{H^{N-1}}\left\|\delta n^{j+1}\right\|_{H^{N-1}}.
\end{aligned}
$$

And the term

$$
\int_{\boldsymbol{R}^3}\partial^k\left(\frac{\nabla n^j\delta u^j}{n^j+\rho_\infty}+\frac{u^j\nabla\delta n^{j+1}}{n^j+\rho_\infty}\right)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k\delta n^{j+1}dx,
$$

is further bounded by

$$
C\left\|\delta u^j\right\|_{H^{N-1}}\left\|\delta n^{j+1}\right\|_{H^{N-1}}+C\left\|\delta n^{j+1}\right\|_{H^{N-1}}^2.
$$

With compensation from (3.15) in (3.14), we have

$$
\begin{aligned}
&\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\partial^k\partial_t\delta u^{j+1}dx\\
&-\int_{\boldsymbol{R}^3}\partial^k\delta u^{j+1}\sum_{i=0}^{k}C_i^k\,\partial^i\delta u^j\partial^{k-i}\nabla\mathrm{u}^j)dx
\end{aligned}
$$

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$$-\int_{R^3} \partial^k \delta u^{j+1} \sum_{i=0}^{k} C_i^k \,(\partial^i u^j \nabla \partial^{k-i} \delta u^{j+1}) dx$$

$$+\lambda \int_{R^3} \partial^k \delta u^{j+1} \partial^k \delta u^{j+1}) dx \;= \mu \int_{R^3} \partial^k \delta u^{j+1} \partial^k \nabla \delta \phi^{j+1}) dx$$
$$- \int_{R^3} \partial^k \delta u^{j+1} \sum_{i=0}^{k} C_i^k \,\partial^i \delta n^j \partial^{k-i} \nabla \mathrm{n}^j) dx$$
$$-\frac{1}{2} \int_{R^3} \frac{\mathrm{P}'(n^j+\rho_\infty)}{(n^j+\rho_\infty)^2} (\partial^k \,\delta n^{j+1})_t^2$$
$$- \int_{R^3} \frac{1}{n^j+\rho_\infty} \partial^k (\nabla u^j \delta n^j) \frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty} \partial^k \delta n^{j+1} dx$$
$$- \int_{R^3} \partial^k \left(\frac{1}{n^j+\rho_\infty}\right) (\nabla u^j \delta n^j) \frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty} \partial^k \delta n^{j+1} dx$$
$$- \int_{R^3} \partial^k \left(\frac{\delta u^j \nabla n^j}{n^j+\rho_\infty} + \frac{u^j \nabla \delta n^{j+1}}{n^j+\rho_\infty} \right) \frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty} \partial^k \delta n^{j+1} dx$$
$$+ \int_{R^3} \partial^k \delta u^{j+1} \,\nabla \frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty} \partial^k \delta n^{j+1} dx$$
$$- \int_{R^3} \partial^k \delta u^{j+1} \sum_{i=1}^{k} C_i^k \,\partial^i \left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty} \right) \partial^{k-i} \nabla \delta n^{j+1} dx.$$

Thus

$$\int_{R^3} \partial^k \delta u^{j+1} \partial^k \partial_t \delta u^{j+1} dx + \lambda \int_{R^3} \partial^k \delta u^{j+1} \delta u^{j+1}) dx$$
$$+\frac{1}{2} \int_{R^3} \frac{\mathrm{P}'(n^j+\rho_\infty)}{(n^j+\rho_\infty)^2} (\partial^k \,\delta n^{j+1})_t^2$$
$$= -\int_{R^3} \frac{\mathrm{P}'(n^j+\rho_\infty)}{(n^j+\rho_\infty)} \partial^k \left(\frac{1}{n^j+\rho_\infty}\right) \partial_t \delta n^{j+1} \partial^k \delta n^{j+1} dx$$

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$$
\begin{aligned}
&-\int_{\boldsymbol{R}^3} \partial^k\left(\frac{1}{n^j+\rho_\infty}\right)(\nabla u^j \delta n^j)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k \delta n^{j+1} dx \\
&+\int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1}\nabla \frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k \delta n^{j+1} dx \\
&-\int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1}\sum_{i=1}^{k} C_i^k\,\partial^i\left(\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\right)\partial^{k-i}\nabla \delta n^{j+1} dx. \\
&-\int_{\boldsymbol{R}^3} \partial^k\left(\frac{\delta u^j \nabla n^j}{n^j+\rho_\infty} + \frac{u^j \nabla \delta n^{j+1}}{n^j+\rho_\infty}\right)\frac{\mathrm{P}'(n^j+\rho_\infty)}{n^j+\rho_\infty}\partial^k \delta n^{j+1} dx \\
&-\int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1}\sum_{i=0}^{k} C_i^k\,\partial^i \delta n^j \partial^{k-i}\nabla \mathrm{n}^j) dx \\
&+\int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1}\sum_{i=0}^{k} C_i^k\,(\partial^i u^j \nabla \partial^{k-i}\delta u^{j+1}) dx \\
&+\int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1}\sum_{i=0}^{k} C_i^k\,\partial^i \delta u^j \partial^{k-i}\nabla \mathrm{u}^j) dx \\
&= \mu \int_{\boldsymbol{R}^3} \partial^k \delta u^{j+1}\partial^k \nabla \delta \phi^{j+1}) dx. \qquad (3.16)
\end{aligned}
$$

Then, after taking the summation over $|K| \leq N-1$ and using the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\delta u^{j+1}\right\|_{H^{N-1}}^2 + \lambda\left\|\delta u^{j+1}\right\|_{H^{N-1}}^2 + C\frac{d}{dt}\left\|\delta n^{j+1}\right\|_{H^{N-1}}^2 \\
&\quad \leq C\left\|\delta n^j\right\|_{H^{N-1}}^2 + C\left\|\delta u^j\right\|_{H^{N-1}}^2 + C\left\|\delta n^{j+1}\right\|_{H^{N-1}}^2 \\
&\quad + C\left\|\delta u^{j+1}\right\|_{H^{N-1}}^2 + C\left\|\nabla \delta \phi^{j+1}\right\|_{H^{N-1}}^2.
\end{aligned}
$$

Therefore

$$
\frac{1}{2}\frac{d}{dt}\left(C\left\|\delta n^{j+1}\right\|_{H^{N-1}}^2 + \left\|\delta u^{j+1}\right\|_{H^{N-1}}^2\right) + \gamma_2\left\|\delta u^{j+1}\right\|_{H^{N-1}}^2
$$

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$$\leq C\left(\left\|\delta n^j\right\|_{H^{N-1}}^2+\left\|\delta u^j\right\|_{H^{N-1}}^2\right)$$
$$+C(\left\|\delta n^{j+1}\right\|_{H^{N-1}}^2+\left\|\delta u^{j+1}\right\|_{H^{N-1}}^2)+C\left\|\nabla\delta\phi^{j+1}\right\|_{H^{N-1}}^2. \quad (3.17)$$

By using the same estimates as before for third equation of (3.13), we have

$$\int_{R^3}\partial^k\delta\phi^{j+1}\partial^k\partial_t\delta\phi^{j+1}dx+\int_{R^3}\partial^k\delta\phi^{j+1}\partial^k\Delta\delta\phi^{j+1}dx$$
$$+(b-a\rho_\infty)\int_{R^3}\partial^k\delta\phi^{j+1}\partial^k\delta\phi^{j+1}dx$$
$$=a\int_{R^3}\partial^k\delta\phi^{j+1}\partial^k(n^j\delta\phi^{j+1})dx$$
$$+a\int_{R^3}\partial^k\delta\phi^{j+1}\partial^k(\phi^j\delta n^j)dx.$$

Then, after taking the summation over $|k|\leq N-1$ and using Holder inequality, the terms on the right side of the previous equation are bounded by

$$C\left\|\delta\phi^{j+1}\right\|_{H^{N-1}}^2+\left\|\delta n^j\right\|_{H^{N-1}}\left\|\delta\phi^{j+1}\right\|_{H^{N-1}}.$$

By using the Cauchy-Schwarz inequality, we get

$$\frac{1}{2}\frac{d}{dt}\left\|\delta\phi^{j+1}\right\|_{H^{N-1}}^2+\left\|\nabla\delta\phi^{j+1}\right\|_{H^{N-1}}^2+\gamma_2\left\|\delta\phi^{j+1}\right\|_{H^{N-1}}^2$$
$$\leq C\left\|\delta\phi^{j+1}\right\|_{H^{N-1}}^2+C\left\|\delta n^j\right\|_{H^{N-1}}^2. \quad (3.18)$$

Taking the linear combination of inequalities (3.17), (3.18), we have

$$\frac{1}{2}\frac{d}{dt}(C\left\|\delta n^{j+1}\right\|_{H^{N-1}}^2+\left\|\delta u^{j+1}\right\|_{H^{N-1}}^2+\left\|\delta\phi^{j+1}\right\|_{H^{N-1}}^2)$$
$$+\gamma_2\left(\left\|\delta u^{j+1}\right\|_{H^{N-1}}^2+\left\|\delta\phi^{j+1}\right\|_{H^{N-1}}^2\right)$$

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$$\leq C\left(\|\delta n^j\|_{H^{N-1}}^2 + \|\delta u^j\|_{H^{N-1}}^2\right)$$
$$+C\left(\|\delta n^{j+1}\|_{H^{N-1}}^2 + \|\delta u^{j+1}\|_{H^{N-1}}^2\right) + \|\delta \phi^{j+1}\|_{H^{N-1}}^2.$$

By using Gronwall's inequality, we obtain

$$\sup_{0\leq t\leq T_1} \|\delta W^{j+1}(t)\|_{H^{N-1}}^2 \leq e^{\int_0^t Cds} \int_0^t \|\delta W^j(s)\|_{H^{N-1}}^2 ds$$
$$+ e^{\int_0^t Cds}\|\delta W^{j+1}(0)\|_{H^{N-1}}^2 \leq CT_1(e^{CT_1}) \sup_{0\leq t\leq T_1} \|\delta W^j\|_{H^{N-1}}^2 .$$

We take $T_1 > 0$ sufficiently small, we find that $(W^j)_{j\geq 0}$is a Cauchy sequence in the Banach Space $C\big([0,T_1]; H^N(R^3)\big)$. Thus, we have the limit function:

$$W = W^0 + \lim_{m\to\infty} \sum_{j=0}^{m} (W^{j+1} - W^{j}),$$

indeed exists in $C\big([0,T_1]; H^{N-1}(R^3)\big)$ and satisfies:

$$\sup_{0\leq t\leq T_1} \|W(t)\|_{H^{N-1}} \leq \sup_{0\leq t\leq T_1} \lim_{j\to\infty} inf \, \|\delta W^j(t)\|_{H^{N-1}} \leq M.$$

Thus, as $j \to \infty$ the limit exists such that:

$$(W^j)_{j\geq 0} \longrightarrow W(t),$$

strongly in $C\big([0,T_1]; H^{N-1}(R^3)\big)$ and as $j' \to \infty$, from step one, we have a subsequence $\{j'\}$ of $\{j\}$, such that:

$$D(W)_{j'\geq 0} \longrightarrow D(W),$$

weakly in $L^2\big([0,T_1]; H^N(R^3)\big)$. Also by same step, we know

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$$(W^j)_{j^{"}\geq 0} \longrightarrow W(t),$$

Weakly in H^N for every fixed $t\epsilon[0,T_1]$, where $j^{"}(t)$ is a subsequence of $\{j'\}$ depending on t. Consequently, we now have a solution $W(t) \in L^\infty\left([0,T_1]; H^N(R^3)\right)$ for the problem (3.1),(3.2).

Step three, we show that $\|W^{j+1}(t)\|_{H^N}^2$ is continuous in time for each $j \geq 0$. For simplicity, let us define the equivalent energy functional:

$$\mathfrak{W}(W^{j+1}(t)) = \|n^{j+1}\|_{H^N}^2 + \|u^{j+1}\|_{H^N}^2 + \|\phi^{j+1}\|_{H^N}^2.$$

Similarly to how we proved (3.11), we have

$$\begin{aligned}
&\left|\mathfrak{W}(W^{j+1}(t)) - \mathfrak{W}(W^{j+1}(s))\right| = \left|\int_s^t \frac{d}{d\theta}\mathfrak{W}(W^{j+1}(\theta))d\theta\right| \\
&\leq C\int_s^t \|W^j(\theta)\|_{H^N}^2 d\theta + C\int_s^t \|n^{j+1}(\theta), \nabla\phi^{j+1}(\theta)\|_{H^N}^2 d\theta \\
&+C\int_s^t \left(1 + \|W^j(\theta)\|_{H^N}^2\right)\|n^{j+1}(\theta), u^{j+1}(\theta), \phi^{j+1}(\theta)\|_{H^N}^2 d\theta \\
&\leq CM^2(s-t) \\
&\quad +C(1+M^2)\int_s^t \|n^{j+1}(\theta), u^{j+1}(\theta), \phi^{j+1}(\theta)\|_{H^N}^2 d\theta \\
&\quad +C\int_s^t \|n^{j+1}(\theta), \nabla\phi^{j+1}(\theta)\|_{H^N}^2 d\theta.
\end{aligned}$$

For any$0 \leq s \leq t \leq T_1$. The time integral on the right-hand side from the above inequality is bounded by (3.12), and therefore $\mathfrak{W}(W^{j+1}(t))$ is continuous in t for each $j \geq 0$. By the same way, we can infer the continuity of $\|n^{j+1}\|_{H^N}^2$, $\|u^{j+1}\|_{H^N}^2$ and $\|\phi^{j+1}\|_{H^N}^2$ in t from (3.8) and (3.9). Hence, $\|W^j(t)\|_{H^N}^2$ is continuous in ***t*** for each $j \geq 1$. Furthermore, $W = (n,u,\phi)$ is a local solution to the Cauchy problem (2.1)–(2.2).

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Finally- the fourth step, in this step, we will prove that there can only be one solution to the Cauchy problem (2.1)-(2.2). Therefore, we suppose that W_1 and W_2 are two local solutions in $C\left([0,T_1];H^N(R^3)\right)$ which satisfy (2.1)-(2.2).

Let $\tilde{n}\,(x,t) = n_1(x,t) - n_2(x,t)$, $\tilde{u}\,(x,t) = u_1(x,t) - u_2(x,t)$, $\tilde{\phi}\,(x,t) = \phi_1(x,t) - \phi_2(x,t)$.

$$\begin{cases} \partial_t\tilde{n} + (n_2+\rho_\infty)\nabla.\tilde{u} = -u_1\nabla\tilde{n} - \tilde{n}\nabla.u_1 - \tilde{u}\nabla n_2, \\ \partial_t\tilde{u} + u_2.\nabla\tilde{u} + \tilde{u}.\nabla u_1 = \mu\nabla\tilde{\phi} - \lambda\tilde{u} - \tilde{n}\nabla n_1 - \dfrac{P'(n_2+\rho_\infty)}{n_2+\rho_\infty}\nabla\tilde{n}, \\ \boldsymbol{\partial_t\tilde{\phi}} = \Delta\tilde{\phi} + (a\rho_\infty - b)\tilde{\phi} + an_2\tilde{\phi} + a\tilde{n}\phi_1. \end{cases} \quad (3.19)$$

By multiplying $\tilde{u}$ to both sides of the second equation of (3.19) and integrating in x, we have

$$\int_{R^3}\tilde{u}\partial_t\tilde{u}dx + \int_{R^3}\tilde{u}u_2.\nabla\tilde{u}dx + \int_{R^3}\tilde{u}\tilde{u}\,.\nabla u_1dx = \mu\int_{R^3}\tilde{u}\,\nabla\tilde{\phi}dx$$

$$-\lambda\int_{R^3}\tilde{u}\tilde{u}dx - \int_{R^3}\tilde{u}\tilde{n}\nabla n_1dx - \int_{R^3}\tilde{u}\,\frac{P'(n_2+\rho_\infty)}{n_2+\rho_\infty}\nabla\tilde{n}dx.$$

Thus

$$\int_{R^3}\tilde{u}\partial_t\tilde{u}dx + \lambda\int_{R^3}\tilde{u}^2dx + \frac{1}{2}\int_{R^3}u_2.\nabla\tilde{u}^2dx + \int_{R^3}\tilde{u}^2\,.\nabla u_1dx$$

$$= \mu\int_{R^3}\tilde{u}\,\nabla\tilde{\phi}dx - \int_{R^3}\tilde{u}\tilde{n}\nabla n_1dx - \int_{R^3}\tilde{u}\left(\frac{P'(n_2+\rho_\infty)}{n_2+\rho_\infty}\right)\nabla\tilde{n}dx.$$

By using integration by parts

$$\int_{R^3}\tilde{u}\partial_t\tilde{u}dx + \lambda\int_{R^3}\tilde{u}^2dx + \frac{1}{2}\int_{R^3}\tilde{u}^2.\nabla u_2dx + \int_{R^3}\tilde{u}^2\,.\nabla u_1dx$$

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$$= \mu \int_{R^3} \tilde{u}\, \nabla\tilde{\phi} dx - \int_{R^3} \tilde{u}\tilde{n}\nabla n_1 dx + \int_{R^3} \nabla\tilde{u}\left(\frac{\mathrm{P}'(n_2+\rho_\infty)}{n_2+\rho_\infty}\right)\tilde{n}dx$$
$$+ \int_{R^3} \tilde{u}\left(\nabla\frac{\mathrm{P}'(n_2+\rho_\infty)}{n_2+\rho_\infty}\right)\tilde{n}dx.$$

With compensation $\left\{\nabla\tilde{u} = -\left(\frac{\partial_t\tilde{n}}{n_2+\rho_\infty} + \frac{u_1\nabla\tilde{n}}{n_2+\rho_\infty} + \frac{\tilde{n}\nabla u_1}{n_2+\rho_\infty} + \frac{\tilde{u}\nabla n_2}{n_2+\rho_\infty}\right)\right\}$ from the first equation of (3.19) and by using integration by parts, and the Cauchy–Schwarz inequality, then we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}\|_{L^2}^2 + \lambda\|\tilde{u}\|_{L^2}^2 + C\frac{d}{dt}\|\tilde{n}\|_{L^2}^2 \le \|\nabla u_1\|_{L^\infty}\|\tilde{u}\|_{L^2}^2$$
$$+\|\nabla u_2\|_{L^\infty}\|\tilde{u}\|_{L^2}^2 + \mu(\|\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{\phi}\|_{L^2}^2)$$
$$+\|\nabla n_1\|_{L^\infty}\left(\|\tilde{n}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2\right) + \|\nabla u_1\|_{L^\infty}\|\tilde{n}\|_{L^2}^2$$
$$+\|\nabla n_2\|_{L^\infty}(\|\tilde{n}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2).$$

Since L^∞ norms of n_i, u_i, ϕ_i where $i = 1, 2$ are bounded, we have:

$$\frac{1}{2}\frac{d}{dt}(\|\tilde{u}\|_{L^2}^2 + C\|\tilde{n}\|_{L^2}^2) + \gamma_1\|\tilde{u}\|_{L^2}^2$$
$$\le C\|\tilde{n}\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}^2 + C\|\nabla\tilde{\phi}\|_{L^2}^2. \qquad (3.20)$$

We have a similar way to estimate $\tilde{\phi}$ as follows:

$$\int_{R^3}\tilde{\phi}\partial_t\tilde{\phi}dx - \int_{R^3}\tilde{\phi}\Delta\tilde{\phi}dx + (b - a\rho_\infty)\int_{R^3}\tilde{\phi}\tilde{\phi}dx$$
$$= a\int_{R^3}\tilde{\phi}n_2\tilde{\phi}dx + a\int_{R^3}\tilde{\phi}\tilde{n}\phi_1 dx.$$

By using integration by parts, and Cauchy–Schwarz inequality, then we have:

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\phi}\|_{L^2}^2 + \frac{1}{2}\|\nabla\tilde{\phi}\|_{L^2}^2 + (b - a\rho_\infty)\|\tilde{\phi}\|_{L^2}^2$$

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$$\leq a\|n_2\|_{L^\infty}\|\tilde{\phi}\|_{L^2}^2 + a\|\phi_1\|_{L^\infty}\left(\|\tilde{n}\|_{L^2}^2 + \|\tilde{\phi}\|_{L^2}^2\right).$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\phi}\|_{L^2}^2 + \gamma_1\|\nabla\tilde{\phi}\|_{L^2}^2 + \gamma_2\|\tilde{\phi}\|_{L^2}^2$$
$$\leq C\|\tilde{\phi}\|_{L^2}^2 + C\|\tilde{n}\|_{L^2}^2. \quad (3.21)$$

By taking a linear combination of inequalities (3.19), (3.20), (3.21), one has:

$$\frac{1}{2}\frac{d}{dt}(\|\tilde{u}\|_{L^2}^2 + C\|\tilde{n}\|_{L^2}^2 + \|\tilde{\phi}\|_{L^2}^2) + \gamma_1\|\nabla\tilde{\phi}\|_{L^2}^2$$
$$+\gamma_2\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{\phi}\|_{L^2}^2\right) \leq C(\|\tilde{n}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{\phi}\|_{L^2}^2).$$

By using Gronwall's inequality, we obtain

$$\sup_{0\leq t\leq T_1}(\|\tilde{n}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{\phi}\|_{L^2}^2)$$
$$\leq e^{\int_0^t C ds}(\|\tilde{n}(0)\|_{L^2}^2 + \|\tilde{u}(0)\|_{L^2}^2 + (\|\tilde{\phi}(0)\|_{L^2}^2).$$

Since the initial data of $(\tilde{n}, \tilde{u}, \tilde{\phi})$ are all zero for $T_1 > 0$. Hence, $W_1 = W_2$ holds. This prove the uniqueness of the local solution and thus completes the proof of Lemma 3.1.

Conclusion

In this paper, we used the energy method to prove the existence and uniqueness the local solution of the compressible chemotaxis system in three dimensions (Cauchy's problem for the hyperbolic angiogenesis model) with the chemoattractant. We divided the proof into four steps and used induction, integral by parts, Cauchy –Schwarz inequality, Holder inequality, and Gronwall's inequality to prove these steps. Thus prove the existence and uniqueness of the local solution.

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