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Terminology and Definitions for School Mathematics Concepts而

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## مصطلحات وتعاريف مفاهيم الريـاضيـــات المدرسيــــة

تألبف

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# Terminology and Definitions for School Mathematics Concepts 

## مصططات وتعاريف مفاهيم الرياضيـــت المدرسيـــة

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## Preface:

The knowledge of terminology is the first step in the Learning of mathematics Language according to Bloom's taxonomy of educational objectives, this is defined in the taxonomy as: " Acknowledge of the reference: For specific verbal and non-verbal symbols ".

This describes the familiarity with the language of mathematics that is the shorthand used to express the ideas verbally and non-verbally.

According to that we indicate that the mathematical concept has three parts (term - definition - examples).

- Term: is the name given to the concept, it can be verbally or non verbally (symbolically or geometrically), such as: Set - vector - equation line.
- Definition: is the statement which describes the relationship between the characteristics of the concept, such as: The triangle is a plane figure with three sides.
- Examples: is the set of elements which have the characteristics of the concept.

We will introduce in this book Some mathematical concepts from different topics pointing the three parts of each with examples, the aim of this is to give the student chance to be familiar with the Mathematical terminology in order to use it in his future studies, these information's were put in a book entitled " school mathematics: concepts and terminology ". This book is intended for university students in the university level. The reason is that Libyan students study mathematics in Arabic language, which does not help them progress in the field of mathematics specially when reading an English mathematics books. This book will provide them with a bank of information's about the terminology in English associated with mathematics concepts. Additionally, this book will contain units which deal with number concepts, geometry concepts, algebra concepts, trigonometric concepts, calculus concepts ... etc. This makes it possible to be used as a text book for the courses in mathematical terminology and school mathematics.

The material in this book is abstracted from a number of school mathematics and university mathematics text books written in English, then arranged and organized and presented using an approach similar to that used by a series of books written in England for the " Saudi Arabia " students entitled " The English for mathematics ".

## Terminology and definitions for school Mathematics concepts

مصطلحات وتعاريف مفاهيم الرياضيات المدرسية

لـغة الكتــــبا اللغة الانجليزيـة.

محتوى الكتـــــبا:

1) يعرض هذا الكتاب عدد من المفاهيم والعلاقات الرياضية مصحوبة بالمصطلحات العلمية

و التعاريف الخاصة لكل منها باللغة الانجليزيـة.
2) (2 يقدم الكتاب عرض الخواص وصفات كل من تلك المفاهيم مع ملاحظات حول كل منها. وزعت مفردات محتوى هذا الكتاب على ثماني وحدات (units) منظمة ومرتبة وفق التقسيم التقليدي لمادة الرياضيات المدرسية بحيث كل وحدة تتنـاول المفاهيم والمصطلحات الرياضية لفر ع من الفرو ع وهي:
(مفاهيم عددية - مفاهيم هندسية - مفاهيم جبرية - مفاهيم مثلثية - مفاهيم هندسية تحليلية ــمفاهيم
متجهات - مفاهيم الحسبان - الحل البياني للمعادلات والمتباينات).
وضع في آخر الكتاب جدول يحوي المصطلحات باللغتين العربية والانجليزية.
نبعت فكرة تأليف هذا الكتاب من خلال الإطلاع على سلسلة من الكتب مماتله مكتوبة في
بريطانيا لطلبة المملكة السعودية.
تم الرجوع الى عدد من كتب الرياضيات المدرسية والجامعية المكتوبة باللغة الانجليزية
لتجميع مادة هذا الكتاب وتحديد المفاهيم الرياضية ومصطلحاتها وتعاريفها المعتمدة. تم الرجوع الى عدد من تقارير الجمعيات الرياضية ومنظمة اليونيسكو حول مفاهيم الرياضيات المدرسية ومداخل عرضها.

أهمية الكتاب:
يزود متعلم الرياضيات ببنك من المعلومات حول مفاهيم الرياضيات المدرسية من حيث
المصطلحات العلمية و التعاريف و الخو اص باللغة الانجليزية.
يساعد المتعلم الذي يدرس الرياضيات باللغة العربية على الإلمام بالمصطلحات والتعاريف
باللغة الانجليزية التي يحتاجها في در استه المستقبلية او عند اطلاعه على كتب اجنبيه.

أ.د. أحمد العريفي الشـارف

# استخذامـات الكتاب: <br> يمكن ان يستخدم كمرجع او كتاب دراسي لدراسة مقرر الرياضيات المدرسبة في كليات <br> التربية و الدراسات العليا. <br> يمكن ان يستخدم كمرجع او كتاب دراسي للراسة مقرر مصطلحات رياضية في كليات <br> العلوم والتنربية والدر اسات العليا. <br> يمكن ان يستخدم كمرجع عند اجر اء البحوث و المشاريع المتحلقة بتربية الرياضبات. 

والسلام عليكم ورحمة الله وبركاته

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\begin{aligned}
& \text { المؤلفان } \\
& \text { أ.د. احمد العريفي الشارف } \\
& \text { أ. عبير خليل صليبي } \\
& \text { كلية النربية جنزور / جامعة طر ابلس }
\end{aligned}
$$

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## Unit.1: Number concepts

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## Unit.1: Number concepts

## Introduction.

Number is a property of any set of objects, but it is not a physical property such as color, shape, size, in fact it is an abstract property which grow out of the idea of classification experiences with the physical world.

The concepts and skills related to number are those concerned with the understanding of sets of numbers and the relationships among them and the ways of representing numbers, and number systems.

Arithmetic is one of the oldest branches of mathematics in fact it is a part of the number topic dealing with the understanding of the meaning of operations on numbers and how they relate to each other and the computational skills.

In this section we will introduce some elementary number concepts.

### 1.1 Numerals:

Numerals are marks, or symbols used to denote numbers.
e. $g: 4,4 \times 1,1+3$ are all numerals for the number (four).
$\frac{1}{2}, \frac{2}{4}, 0.5, \frac{1}{4}+\frac{1}{4}$ are all numerals for the number one - half.
Note: The symbols used most widely for numbers are called Hindu-Arabic numerals this system is made up of ten.

Basic numerals called digits.
$0,1,2,3,4,5,6,7,8,9$.
The Roman numerals are marks as I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII.

### 1.2 Ordinal number:

The ordinal number is a number which indicates the position of an object in the set when the objects were arranged and ordered in some way, the ordinals are written as:

First (1st) - Second (2nd) - Third (3rd) - Fourth (4th) - Fifth (5th) -Sixth (6th) - Seventh (7th) - Eighth (8th) - Ninth (9th) -Tenth (10th).

### 1.3 Cardinal number:

The cardinal number is a number which indicates how many elements in a set.

## Examples:

The numerals are used to represent ordinals such as in the figure (1.1):


0
Zero


1
One

2
Two


3
Three


4
Four


5
Five


6
Six


Nine


7
Seven


Ten

Fig.(1.1)

### 1.4 Number sets:

Different systems of numbers have been discovered in different times according to man needs.

The first numbers people started using were the" counting numbers, which the primitive man used to count his belongings, this numbers are also called natural numbers $1,2,3,4, \ldots$ It took some time for zero to be invented By Arab mathematicians and when added to the set of natural numbers this set then called the whole numbers $0,1,2,3,4,5, \ldots$ and time by time a new set of numbers appears to use. we will list the sets of numbers according to their appearance in man's life as follows:

## 1- Natural numbers $N$ :

These are numbers which are positive integers with 0 (zero) excluded.
$N=\{1,2,3,4, \ldots\}$

## 2- Whole numbers $\mathbf{W}$ :

These are natural numbers with 0 (zero) included.

$$
W=\{0,1,2,3,4, \ldots\}
$$

## 3- Integer I:

These are negative and positive whole numbers. But 0 (zero) is neither positive nor negative - it is just an integer.

$$
I=\{\ldots,-4,-3,-2,-1,0,+1,+2,+3,+4, \ldots\}
$$

## 4- Rational numbers Q:

These are numbers which can be expressed as a fraction or a ratio, such as

$$
\begin{aligned}
& Q=\left\{\frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 0.8, \frac{4}{1}, \frac{19}{23}, \ldots\right\} \\
& 0.8=0.8=\frac{8}{10}=\frac{4}{5} \quad: \frac{4}{1}=4
\end{aligned}
$$

## 5- Irrational numbers $\mathbf{Q}^{\prime}$ :

These are numbers which cannot be expressed as fractions or ratios such as: $\pi, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$, for Instance, $\pi=3.1415926 \ldots$ to seven decimal places, it has been calculated to thousands of decimal places, but the decimal has never ended.

## 6- Real numbers R:

This is a set which is made up of all sets of numbers mentioned earlier (rational numbers and irrational numbers). This means that:
$N \subset W \subset I \subset Q \subset R \supset Q^{\prime}$. see the figure (1.2)


Fig.(1.2)
Some other subsets of the sets of numbers are:
Odd numbers
$O D=\{1,3,5,7, \ldots\}$
Numbers which will not divide by 2 exactly.

## Even numbers

$E V=\{0,2,4,6, \ldots\}$
Numbers which will divide by 2 exactly.

## Prime numbers

$P R=\{2,3,5,7,11, \ldots\}$
A prime number is a whole number which can only be divided by itself and 1 , (0 and 1) are not considered to be prime numbers.

## Square numbers

These are formed by multiplying a whole number by itself.
Example: $5 \times 5=25$, so 25 is a square number. We often write $5 \times 5$ as $5^{2}$ (five squared). Hence the set of square numbers will be $\left\{1^{2}, 2^{2}, 3^{2}, \ldots\right\}$ i.e., $\{1,4,9, \ldots\}$ In diagram form, square numbers can be represented by dots in the form of a square as shown in the figure (1.3):


Fig.(1.3)

## Cubic numbers:

$\left\{1^{3}, 2^{3}, 3^{3}, \ldots\right\}$ i.e $\{1,8,27,64, \ldots\}$

## Triangular numbers:

$$
\{1,3,6,10, \ldots .\}
$$

These numbers can be arranged in dot form to show how equilateral
Triangles are formed as shown in the figure (1.4).


Fig.(1.4)

## Square root:

This is the reverse operation of squaring (above), the symbol used is $\sqrt{ }$ and hence $\sqrt{25}=5, \sqrt{100}=10$ etc.

## Factors:

If a number divides exactly into another number (i.e., without leaving a remainder) the first number is called a FACTOR of the second.

Example: 2 is a factor of 8 (since $8 \div 2=4$, a whole number).

## Multiples:

If a number divides exactly into another number, the second number is called a MULTIPLE of the first. Example: 8 is a multiple of $2 ; 15$ is a multiple of 3; etc.

## Highest common (HCF):

The HCF of two or more numbers is the largest number which will divide into them exactly (i.e., without leaving a remainder).
Example: The HCF of 72 and 96

$$
72=3 \times 24 \quad, 96=4 \times 24
$$

Hence 24 is the HCF of 72 and 96 .

## Indices:

When we multiply a row of numbers which are all the same value, we get a POWER of the number.
Example: $9 \times 9 \times 9 \times 9 \times 9$ is the fifth power of 9 and in index form is written as $9^{5}$.
If the index is negative, i.e. $a^{-m}$ then this can be rewritten as $\frac{1}{a^{m}}$.
e.g. $\quad 4^{-2}=\frac{1}{4^{2}}=\frac{1}{16}$

If the index is fractional, i.e. $\mathrm{a}^{1 / \mathrm{m}}$ then this can be rewritten as $\sqrt[m]{a}$
For example, $49^{\frac{1}{2}}=\sqrt{49}=7$ or $125^{\frac{1}{3}}=\sqrt[3]{125}=5$.

## Lowest common multiple (LCM):

The LCM of two or more numbers is the smallest number which is exactly divisible by them. Example: the LCM of 4, 8, 20
Factors of $4=2 \times 2=2^{2}$
Factors of $8=2 \times 2 \times 2=2^{3}$
Factors of $20=2 \times 2 \times 5=2^{2} \times 5$
Hence the LCM of $4,8,20=2^{3} \times 5=40$

### 1.5 Number operations:

The four operations on numbers, (addition-subtraction-multiplicationdivision), sometimes people call them the arithmetic rules, have a
relationship with the operations on sets which used to introduce the meaning of each these number operations to young children as follows:

## Addition:

Addition of numbers does not mean to increase but to group, join or rename a pair of numbers as a single number.
So, the operation
$3+2=5$ means that the number $3+2$ is the same as 5 .
This means that there is a relationship between the operation of addition of numbers and the operation of union of sets.
So, the above operation $3+2=5$ can be represented geometrically as shown in the figure (1.5):


Fig.(1.5)

This operation is read verbally as:
Three added to two equals five.
Or three plus two equal five, or three and two equals five, in vertical form put as:


## Subtraction:

Subtraction is sometimes referred to as an operation distinct from addition. More often, however, subtraction is referred to as the inverse of addition. Conventionally, addition has been defined as a joining operation such as $3+2=5$ is defined as:
3 joined with 2 produces or gives 5 . subtraction as its inverse would be separating operation such as $5=3+$ ? this means that subtraction may be thought of as finding the missing addend when the sum and one addend are known as $5=3+\square$ or $3+\square=5$

And written as $5-3=2$, this operation can be represented geometrically as shown in the figure (1.6):

5 5


3

2

Fig.(1.6)


This operation is read verbally as:
Five take away three equal two
Or
Five minus three equal two
Or
Three subtracted from five equal two.

## Multiplication:

Multiplication operation is defined as repeated addition operation.

That is an operation as $3 \times 2=6$ can be interpreted as 3 sets each has two elements joined give set of 6 elements, and this can be put in a short form as 3 sets of 2 is 6 , this operation can be represented geometrically as shown in the figure (1.7):


Fig.(1.7)
This shows that in the multiplication operation the first number (multiplier) is the number of sets and the second number (multiplied) is the number of elements in each set, and the third number (product) is the number of elements in the resulting set, we notice that the repetition is to the second number (multiplied) not to the first number, So the operation


Is read verbally as:
3 sets of 4 equals 12
Or 3 times of 4 equals 12
Or 3 multiplied by 4 equals 12
This operation can be represented geometrically as shown in the figure (1.8):


Fig.(1.8)

This shows that $3 \times 4=4+4+4$
That is 4 repeated 3 times gives 12
$3 \times 4$ does not mean $3+3+3+3$ however it equals 12 this is due to $3 \times 4=4 \times 3$ according to the commutativity of this operation which means that:

$$
\begin{aligned}
& 3 \times 4=4+4+4=12 \\
& 4 \times 3=3+3+3+3=12
\end{aligned}
$$

## Division:

Multiplication and division are referred to as inverse operations that is if $a \times b=c$, then $c \div b=a$, so since multiplication is a repeated addition the division then is a separating operation, that is an operation such as $12 \div 3=4$ interpreted as a set of 12 elements divided (separated) into 3 equal subsets (parts) the result was 4 elements in each part (subset)
This operation can be represented geometrically as shown in the figure (1.9):


The operation $12 \div 3=4$ is sometimes put as:


And read verbally as:
12 divided by 3 equals 4
or
3 in to 12 is 4

### 1.6 Properties of real numbers:

We will list some fundamental (basic) properties of the set of real numbers ( R ) under the ordinary operations of addition (+) and multiplication $(\times)$, these basic properties apply for all the mentioned subsets of (R):
If $a, b, c$ are any real numbers $a, b, c \in \mathrm{R}$ then the following are true:

## Closure property

$a+b \in \mathrm{R}$
$a \times b \in \mathrm{R}$
This indicates that the set of real numbers is closed under addition and multiplication operations.
Example $3,2 \in \mathrm{R} \rightarrow 3+2=5 \in \mathrm{R}, 3 \times 2=6 \in \mathrm{R}$

## Commutative property:

$a+b=b+a$
$a \times b=b \times a$ or $a b=b a$.
This indicates that the operations of addition and multiplications are commutative with respect to (R), and this is called commutatively or commutative property or commutative law.
Example $: 4,5 \in \mathrm{R} \longrightarrow 4+5=5+4=9, \quad 4 \times 5=5 \times 4=20$

## Associative property:

$a+(b+c)=(a+b)+c$
$a \times(b \times c)=(a \times b) \times c$ or $a(b c)=(a b) c$
This indicates that the operations of addition and multiplications are associative with respect to ( R ), and this is called associativity or associative property or associative law.

## Example:

$2,3,4 \in \mathrm{R} \longrightarrow 2+(3+4)=(2+3)+4$,
$2 \times(3 \times 4)=(2 \times 3) \times 4$ or $2(3 \times 4)=(2 \times 3) 4$

## 1- Distributive property:

i) $a(b+c)=a \times b+a \times c$ or

$$
\begin{array}{ll}
a(b+c)=a b+a c & \text { or } \\
(b+c) \mathrm{a}=\mathrm{ba}+\mathrm{ca} & \text { or } \\
a(b-c)=a b-a c &
\end{array}
$$

ii) but $a+(b \times c) \neq(a+b) \times(a+c)$

This indicates that multiplication is distributive over addition and subtraction but addition and subtraction are not distributive over multiplication.
This property is called distributivity or distributive property or distributive law.
Example:

$3,4,5 \in \mathrm{R}$$\longrightarrow$| $3(4+5)=3$ | $\times 4+3 \times 5$ |
| ---: | :--- |
| $(4+5) 3$ | $=4 \times 3+5 \times 3$ |
| $3(5-4)$ | $=3 \times 5-3 \times 4$ |
| But | $3+(5 \times 4)$ $\neq(3+5) \times(3+4)$ |

### 1.7 Additional properties of numbers:

We will list here some additional properties which are not included in the basic properties list because some of them do not apply to some subsets of the real numbers however they do apply to the set of real number as a whole.

## Additive identity property:

$$
\begin{aligned}
a+0 & =0+a \\
& =a
\end{aligned}
$$

Zero is called the additive identity

## Multiplicative identity property:

$$
\begin{aligned}
a \times 1 & =1 \times a \\
& =a
\end{aligned}
$$

One is called the multiplicative identity.

## Additive inverse property:

$$
\begin{aligned}
a+(-a) & =-a+a \\
& =0
\end{aligned}
$$

$a$ and $-a$ are called additive inverses of each other

## Multiplicative inverse property:

$a \times \frac{1}{a}=\frac{1}{a} \times a ; a \neq 0=1$
$\frac{\mathbf{1}}{\boldsymbol{a}}$ is the multiplicative inverse of $a$, which is called also its reciprocal.

## Division properties:

$$
a \div b=a \times \frac{1}{b}=\frac{a}{b} ; b \neq 0
$$

## Equality properties:

i) $a=b \longleftrightarrow a+c=b+c$ $a c=b c$
ii) $a b=0 \longleftrightarrow a=0$ or $b=0$

## In equality properties:

i) $a>b \longleftrightarrow a-b>0 ; a+c>b+c$
$a<b \longleftrightarrow a-b<0 ; a+c<b+c$
ii) $a>b \longleftrightarrow\left\{\begin{array}{l}a c>b c ; c>0 \\ a c<b c ; c<0\end{array}\right.$

$$
a<b \longleftrightarrow\left\{\begin{array}{l}
a c<b c \quad ; c>0 \\
a c>b c ; c<0
\end{array}\right.
$$

## 1-8 Fractions

Fraction is a mathematical expression denoted by a numeral in the form of a quotient of two integers $a, b$ such as $\frac{a}{b} ; b \neq 0$ representing the relationship between the total number of parts of the whole and the number of parts taken of it as shown in the figure ( $1.10-\mathrm{a}, \mathrm{b}, \mathrm{c}$ ).

## Examples:

i)


The whole

Divided
in to 4 parts

4 equal parts


Fig.(1.10-a)
أ.د. أحمـد العريـفي الشــارف خلـيل صليبي

The shaded part of this figure is $\frac{3}{4}$ of the whole.
ii)


The shaded part of this figure is $\frac{2}{5}$ of the whole.

Fig.(1.10-b)
iii)


The number of black balls is $\frac{4}{9}$ of the whole number

Fig.(1.10-c)

### 1.9 Sub-fractional concepts

We will list here some sub concepts related to fractions.

## Numerator and denominator:



The denominator is the total number pants in the whole.
The numerator is the number of parts shaded (taken) from the whole.

## Equivalent fractions:

We can see that $\frac{\mathbf{3}}{\mathbf{4}}=\frac{\mathbf{6}}{\mathbf{8}}$ and these are known as EQUIVALENT fraction

> حقوق الطبع محفوظة لمركز العلوم و التّقتية للبحوث و الدراسات
$\frac{\mathbf{3}}{\mathbf{4}}$ is in fact $\frac{\mathbf{6}}{\mathbf{8}}$ in its lowest terms(see the figure(1.11))


$\frac{6}{8}$

Fig.(1.11)
Other fractions equivalent to $\frac{\mathbf{3}}{4}$ include $\frac{9}{12}, \frac{\mathbf{1 2}}{16}, \frac{15}{20}$ etc.

## Proper (vulgar) fractions

If the numerator of a fraction is smaller than the denominator, the fraction is called a VULGAR or proper fraction.
Example $\quad \frac{2}{3}$ is a vulgar fraction.

## Improper fractions

If the numerator of a fraction is larger than the denominator, the fraction is called IMPROPER.
Example: $\quad \frac{\mathbf{8}}{\mathbf{5}}$ is an improper fraction.

## Mixed number

If a number consists partly of an integer and partly of a fraction, it is called a MIXED NUMBER.
Example: $2 \frac{3}{5}=2+\frac{3}{5}$ is a mixed number.

## Common multiple (C.M)

The common multiple for a set of number is the number which can be divided exactly by each number in this set.
Note:
i) The least common multiple or lowest common multiple (L.C.M) is the smallest common multiple between the common multiples of a set of numbers.
ii) In the case of fractions denominators, the (L.C.M) is called the least common denominator (L.C.M).

## Examples:

i) For the numbers 2, 3, 4, 6 the (C.M) is 12 or 24 , or $36, \ldots$

The (L.C.M) is 12
ii) For the fractions $\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{3}}{\mathbf{4}}, \frac{\mathbf{2}}{\mathbf{5}}$ The (L.C.M) is 20 .

## Greatest common divisor (G C D):

The greatest common divisor (G C D) or highest common factor (H C F) of a set of numbers is the largest number which divides exactly each of these numbers.

## Examples:

i) The G C D of 12 and 28 is 4.
ii) The G C D of $15,45,60$ is 15 .
iii) The H C F of $42,60,84$ is 6 .

### 1.10 Operations on fractions:

The four operations on fractions (addition - subtraction - multiplication and division) are defined as follows:

## Addition:

Only fractions having the same denominators may be added .so if the denominators are not the same then they must be made so by finding their common denominator, preferably the (L. C. D) and then rewriting all with the (L. C. D) and then adding the numerators. Also note that for adding mixed numbers the integers added first then fractions.
The formula for addition is
i) $\frac{a}{\boldsymbol{b}}+\frac{\boldsymbol{c}}{\boldsymbol{b}}=\frac{\boldsymbol{a}+\boldsymbol{c}}{\boldsymbol{b}}$ in case of the same denominator
ii) $\frac{\boldsymbol{a}}{\boldsymbol{b}}+\frac{\boldsymbol{c}}{\boldsymbol{d}}$ denominators are not the same we can use (b . d) as a possible common denominator however if it is not the (L.C.M) then:
$\frac{a}{b}=\frac{a}{b} \cdot \frac{d}{d}=\frac{a . d}{b . d}$
$\frac{c}{d}=\frac{c}{d} \cdot \frac{b}{b}=\frac{c . b}{d . b}$
$\frac{a}{b}+\frac{c}{d}=\frac{a . d}{b . d}+\frac{b . c}{b . d}=\frac{a . d+b . c}{b . d}$
$\frac{\boldsymbol{a}}{\boldsymbol{b}}+\frac{\boldsymbol{c}}{\boldsymbol{d}}=\frac{\boldsymbol{a} \cdot \boldsymbol{d}+\boldsymbol{b} \cdot \boldsymbol{c}}{\boldsymbol{b} \cdot \boldsymbol{d}} \quad$ in case of different denominators.

## Note:

i) To convert the improper fraction $\frac{\boldsymbol{a}}{\boldsymbol{b}}$ to a mixed number, divide $a$ by $b$, giving a whole number and a remainder hence the mixed number is represented using these two numbers in the form $\boldsymbol{a}_{\mathbf{1}} \frac{\boldsymbol{a}_{2}}{\boldsymbol{b}}$ which means $\left(a_{1}+\frac{a_{2}}{b}\right)$.
ii) To convert the mixed number $a_{1} \frac{a_{2}}{b}$ to an improper fraction write it in the form $\left(\boldsymbol{a}_{1}+\frac{\boldsymbol{a}_{2}}{\boldsymbol{b}}\right)$ and then use the idea of addition of fraction to put it in the form $\frac{a}{b}$.

## Examples:

i) $\frac{\mathbf{2}}{\mathbf{5}}+\frac{\mathbf{1}}{\mathbf{5}}=\frac{\mathbf{2 + 1}}{5}=\frac{\mathbf{3}}{5}$ (the same denominator)
ii) $\frac{3}{4}+\frac{2}{3}=\frac{3.3+4.2}{4.3}=\frac{9+8}{12}=\frac{17}{12}=1 \frac{5}{12}$
iii) $2 \frac{3}{4}+4 \frac{7}{12}=(2+4)+\left(\frac{3}{4}+\frac{7}{12}\right)=6+\frac{9}{12}+\frac{7}{12}$

$$
=6+\frac{16}{12}=6 \frac{16}{12}=7 \frac{1}{3}
$$

iv) $\frac{19}{8}=2$ and remainder 3
$\therefore \frac{19}{8}=2+\frac{3}{8}=2 \frac{3}{8}$
v) $3 \frac{2}{7}=3+\frac{2}{7}$

$$
=\frac{3}{1}+\frac{2}{7}=\frac{21}{7}+\frac{2}{7}=\frac{23}{7}
$$

## 2- Subtraction:

Subtracting fractions works exactly like adding fractions, this means that fractions can be subtracted by rewriting them with a common denominator. The formula for subtraction is:
i)

$$
\frac{a}{b}-\frac{c}{b}=\frac{\boldsymbol{a}-\boldsymbol{c}}{\boldsymbol{b}} \quad \text { in case of the same denominator. }
$$

ii)

$$
\frac{a}{b}-\frac{c}{d}=\frac{a \cdot d-b . c}{b \cdot d} \text { in case of different denominators. }
$$

## Examples:

i)

$$
\frac{5}{7}-\frac{3}{7}=\frac{5-3}{7}=\frac{2}{7} \text { (the same denominator) }
$$

ii) $\frac{3}{4}-\frac{5}{7}=\frac{3.7-4.5}{4.7}=\frac{21-20}{28}=\frac{1}{28}$
iii) $4 \frac{7}{8}-2 \frac{5}{6}=\left(4+\frac{7}{8}\right)-\left(2+\frac{5}{6}\right)=4+\frac{7}{8}-2-\frac{5}{6}$

$$
=4-2+\left(\frac{7}{8}-\frac{5}{6}\right)
$$

$$
=2+\frac{7.6-8.5}{8.6}
$$

$$
=2+\frac{42-40}{48}
$$

$$
=2+\frac{2}{48}=2 \frac{1}{24}
$$

## 3- Multiplication:

To multiply a fraction by a fraction or by a whole number, we must multiply the numerators together, and then multiply the denominators together.

The formula for multiplication is:

$$
\frac{a}{b} \cdot \frac{c}{d}=\frac{a \cdot c}{b \cdot d}=\frac{a c}{b d}
$$

## Notes:

i) Notice that in multiplication the denominators do not need to be the same.
ii) Putting a fraction in lowest terms can be done by a conciliation method which is the rule of multiplication written back words.
iii) To multiply mixed numbers, convert them to improper fractions before the multiplications are done.

## Examples:

i) $\frac{1}{2} \cdot \frac{3}{4}=\frac{1.3}{2.4}=\frac{3}{8}$
ii) $2 \cdot \frac{3}{7}=\frac{2}{1} \cdot \frac{3}{7}=\frac{2.3}{1.7}=\frac{6}{7}$
iii) $\frac{3}{6}=\frac{3.1}{3.2}=\frac{3}{3} \cdot \frac{1}{2}=\frac{1}{2}$ (in the lowest term)

This can be done by cancelling a 3 from the top and bottom of the fraction:
This is usually written as: $\frac{\frac{3}{6}}{6}=\frac{\mathbf{1}}{\mathbf{2}}$
iv) $\frac{120}{-80}=\frac{12}{-8}=\frac{3}{2}=1 \frac{1}{2}$ (in the lowest term)
v) $5 \frac{1}{3} \cdot 2 \frac{1}{4}=\frac{16}{3} \cdot \frac{9}{4}=\frac{144}{12}=12$

## 4-Division:

In dividing fractions, we use a property of fractions called the golden rule of fractions which indicates that you can multiply the top and bottom of a fraction by the same real number except zero, that is $\frac{\boldsymbol{a}}{\boldsymbol{b}}=\frac{\boldsymbol{a} \cdot \boldsymbol{c}}{\boldsymbol{b} \cdot \boldsymbol{c}}$.

This indicates that to divide two fractions multiply top and bottom by the reciprocal of the denominator.

## Note:

The reciprocal of a number is a number, which when multiplied by each other gives 1 , this is in fact the multiplicative inverse of the number, so that.
$a \cdot \frac{1}{a}=1, \frac{a}{b} \cdot \frac{b}{a}=1$, then in dividing fraction we have the following cases:
i. $\frac{a}{\frac{b}{c}}=\frac{a \cdot \frac{c}{b}}{\frac{b}{c} \cdot \frac{c}{b}}=\frac{a \cdot \frac{c}{b}}{1}=\frac{a . c}{b}$
ii. $\quad \frac{\frac{a}{b}}{c}=\frac{\frac{a}{b} \cdot \frac{1}{c}}{c \cdot \frac{1}{c}}=\frac{\frac{a}{b . c}}{1}=\frac{a}{b . c}$
iii. $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}}=\frac{a . d}{a . c}=\frac{a . d}{b . c}$

Therefore, this formula can be written as:
i) $\frac{a}{\frac{b}{c}}=\frac{a . c}{b}$
ii) $\frac{\frac{a}{b}}{c}=\frac{a}{b c}$
iii) $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a . d}{b . c}$

Note: The formula for division can be put in another form by converting division in to multiplication by using the idea of reciprocal as follows:
i)

$$
a \div \frac{b}{c}=\frac{a}{\frac{b}{c}}=\frac{a . c}{b}=a \times \frac{c}{b}
$$

ii)

$$
\frac{a}{b} \div c=\frac{\frac{a}{b}}{c}=\frac{a}{b . c}=\frac{a}{b} \times \frac{1}{c}
$$

iii)

$$
\frac{a}{b} \div \frac{c}{d}=\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b} \times \frac{d}{c}
$$

There for these formulae can be written as:
i)

$$
a \div \frac{b}{c}=a \times \frac{c}{b}
$$

ii)

$$
\frac{a}{b} \div c=\frac{a}{b} \times \frac{1}{c}
$$

iii)

$$
\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \times \frac{d}{c}
$$

### 1.11 Decimal fractions:

Decimal fractions (also called decimals) are fractions whose denominators are a power of ten and for which there is a convention of
writing them in such a way without a denominator using a dot (.) called decimal point to separate the whole number from the fractional parts.

## Examples:

i)

$$
\begin{array}{ll}
\frac{\mathbf{2}}{\mathbf{1 0}}=\mathbf{0 . 2} & \text { read out as } 0 \text { point two } \\
\frac{\mathbf{3}}{\mathbf{1 0 0}}=\mathbf{0 . 0 3} & \text { read out as } 0 \text { point } 0 \text { three }
\end{array}
$$

ii)
iii)

$$
\frac{\mathbf{5}}{\mathbf{1 0 0 0}}=0.005 \text { read out as } 0 \text { point } 00 \text { five }
$$

## Notes:

1- Changing fractions to decimals.
To change a fraction into a decimal we divide the NUMERATOR by the DENOMINATOR.

Example $1: \frac{\mathbf{3}}{\mathbf{4}}=0.75 ; \frac{\mathbf{4}}{\mathbf{5}}=0.8$
Example $2: \frac{21}{25}=0.84$
2- Changing decimals to fractions.
To change a decimal into a fraction, we must see how many decimal places are involved and act accordingly.
Example 1: 0.5 (one decimal place - denominator 10) then

$$
0.5=\frac{\mathbf{5}}{\mathbf{1 0}}=\frac{\mathbf{1}}{\mathbf{2}}(\text { in lowest terms })
$$

Example 2: 0.625 (three decimal places - denominator 1000), then

$$
0.625=\frac{\mathbf{6 2 5}}{\mathbf{1 0 0 0}}=\frac{\mathbf{5}}{\mathbf{8}}(\text { in lowest terms }) .
$$

Examples: convert each of the following fraction to a decimal.

$$
\frac{25}{10}=\frac{20+5}{10}=\frac{20}{10}+\frac{5}{10}=2+0.5=2.5
$$

and read out as two point five

$$
\text { 2- } \begin{aligned}
& \quad \frac{\mathbf{3 2 0 6}}{\mathbf{1 0 0}}=\frac{\mathbf{3 0 0 0}+\mathbf{2 0 0}+\mathbf{0}+\mathbf{6}}{\mathbf{1 0 0}}=\frac{\mathbf{3 0 0 0}}{\mathbf{1 0 0}}+\frac{\mathbf{2 0 0}}{\mathbf{1 0 0}}+\frac{\mathbf{0}}{\mathbf{1 0 0}}+\frac{\mathbf{6}}{\mathbf{1 0 0}} \\
& =30+2+0+0.06 \\
& =32.06
\end{aligned}
$$

And read out as thirty-two point 0 six.
3- $\frac{213}{1000}=\frac{200+10+3}{1000}=\frac{200}{1000}+\frac{10}{1000}+\frac{3}{1000}$

$$
\begin{aligned}
& =0.2+0.01+0.003 \\
& =0.213
\end{aligned}
$$

And read out as 0 point two one three
According to the above examples we can write directly as:
$\frac{\mathbf{5 2 5}}{\mathbf{1 0 0 0 0}}=0.0525, \frac{\mathbf{1 2 0 5}}{\mathbf{1 0 0}}=12.05$
$\frac{7325}{1000}=7.325, \frac{12001}{1000}=12.001$

Examples: convert each of the following decimals into a fraction
1- $12.75=12+0.75=\frac{\mathbf{1 2 0 0}}{\mathbf{1 0 0}}+\frac{\mathbf{7 0}}{\mathbf{1 0 0}}+\frac{\mathbf{5}}{\mathbf{1 0 0}}=\frac{\mathbf{1 2 0 0}+\mathbf{7 5}}{\mathbf{1 0 0}}=\frac{\mathbf{1 2 7 5}}{\mathbf{1 0 0}}$
2- $4.325=\frac{\mathbf{4 0 0 0}}{\mathbf{1 0 0 0}}+\frac{\mathbf{3 0 0}}{\mathbf{1 0 0 0}}+\frac{\mathbf{2 0}}{\mathbf{1 0 0 0}}+\frac{5}{\mathbf{1 0 0 0}}=\frac{4000+300+20+5}{\mathbf{1 0 0 0}}$
$=\frac{4325}{1000}$

## Note:

According to the above examples we can write directly as:
$32.5=\frac{\mathbf{3 2 5}}{\mathbf{1 0}}, 12.56=\frac{\mathbf{1 2 5 6}}{\mathbf{1 0 0}}$
$0.475=\frac{\mathbf{4 7 5}}{\mathbf{1 0 0 0}}, 0.023=\frac{\mathbf{2 3}}{\mathbf{1 0 0 0}}$

### 1.12 operations on decimals:

## Addition / subtraction:

Remember that, when adding and subtracting in decimal form, it is helpful if we work in vertical columns and the decimal points must be under one another. the addition / subtraction is then carried out in exactly the same way as for the ordinary number system.
Example: $0.328+7.63+6$

|  | $\mathbf{0 . 3 2 8}$ |
| :---: | :---: |
| $\mathbf{7 . 6 3 0}$ |  |
| Hence: | $\mathbf{6 . 0 0 0}$ |
|  | $\mathbf{1 3 . 9 5 8}$ |

## 2- Multiplication:

To carry out this operation we ignore the decimal points and multiply as for the ordinary number system, placing the decimal point in its correct position afterwards.

Example: $14.7 \times 0.59$ is written as:

$$
\begin{array}{r}
147 \\
\times 59 \\
\hline=1323 \\
+7350 \\
\hline=8673
\end{array}
$$

We can say that the answer will have as many decimal places (numbers after the decimal point) as the number of decimal places in the two numbers we are multiplying, added together.

In the above case $1+2=3$ decimal places required.
Hence $14.7 \times 0.59=8.673$

## 3-Division:

In order to do this operation, we must always have the DIVISOR (the number we are dividing by it) as a whole number.

Example: $17.04 \div 0.8$ in this case we need the divisor as 8 , not 0.8 .
To do this we multiply 0.8 by 10, but also, we have to multiply 17.04 by 10 giving 170.4. The problem now becomes $170.4 \div 8$ which can be dealt with as for ordinary short division, i.e.


Hence $17.04 \div 0.8=21.3$

### 1.13 Standard form or scientific notation:

A number expressed in the form $A \times 10^{n}$, where $A$ is a number between 1 and 10 and $n$ is a positive or negative, is writing in scientific notation or STANDARD FORM. It is particularly useful for expressing very large or small numbers.

Example1: $5300000000=5.3 \times 10^{9}$ in standard form. To obtain this:

1- Place the decimal point between the first and second numbers so chat $A=5.3$.
2- Count the number of decimal place moves to the right required to restore the decimal point to its original position. It is 9 . Therefore 5.3 is multiplied by $10^{9}$. Hence the number $=5.3 \times 10^{9}$.
Example 2: The very small number $0.000000054=5.4 \times 10^{-8}$. To obtain this:

1- Proceed as before to make $A=5.4$
2- To restore the decimal point to its original position it must be moved 8 places to the left. 5.4 is divided by $10^{8}$ or multiplied by $10^{-8}$
Hence the number $=5.4 \times 10^{-8}$.
Note: In standard form, numbers less than one always have a negative power of 10 .

## Examples:

Put each of the following numbers in scientific notation (standard form):
i) $30000=3 \times 10000=3 \times 10^{4}$
ii) $3150000=315 \times 10^{4}=\frac{315}{10^{2}} \times 10^{2} \times 10^{4}$

$$
=3.15 \times 10^{6}
$$

iii) $51623=\frac{51623}{10^{4}} \times 10^{4}=5.1623 \times 10^{4}$
iv) $805163=\frac{805163}{10^{5}} \times 10^{5}=8.05163 \times 10^{5}$
v) $0.015=\frac{15}{10^{3}}=\frac{\frac{15}{10}}{10^{3}} \times 10$

$$
=\frac{1.5}{10^{3}} \times 10=\frac{1.5}{10}
$$

$$
=1.5 \times 10^{-2}
$$

vi) $0.00000128=\frac{128}{10}=\frac{\frac{128}{10^{2}}}{10^{8}} \times 10^{2}$

$$
=\frac{1.28}{10} \times 10^{2}=1.28 \times 10^{-6}
$$

vii) $0.0751=\frac{751}{104}=\frac{\frac{751}{10^{2}}}{10^{4}} \times 10^{2}$

$$
=\frac{7.51}{10^{4}} \times 10^{2}=7.51 \times 10^{-2}
$$

Put each of the following numbers in decimal notation.
viii) $3.52 \times 106=\frac{352}{10^{2}} \times 106$

$$
=352 \times 104=3520000
$$

ix) $2.3564 \times 104=\frac{23564}{10^{4}} \times 104=23564$
x) $8.1 \times 10-9=\frac{81}{10} \times 10-9$
$=81 \times 10-10=0.0000000081$
xi) $5.213 \times 10-5=\frac{5213}{10^{3}} \times 10-5$

$$
\begin{aligned}
& =5213 \times 10-8 \\
& =0.00005213
\end{aligned}
$$

### 1.14 Percentages:

PERCENTAGE means part of a hundred. percentages are simply fractions with a denominator of 100 . We usually use percentages when we talk about examination marks at school or about money involving discounts, profits etc. For instance, 50 percent ( $50 \%$ ) means 50 parts of a hundred parts, which can be written as a fraction $\frac{50}{100}$ or $\frac{1}{2}$ (in lowest terms).

## Notes:

1- Changing percentages to fractions /decimals
Example: $20 \%=\frac{20}{100}=\frac{1}{5}$ in lowest terms $=0.2$ in decimal form.
2- Changing fractions/decimals to percentages
In order to do this, we simply multiply the fraction/decimal by 100.
Example 1: $\frac{4}{5}=\frac{4}{5} \times 100 \%=\frac{400 \%}{5}=80 \%$
Example 2: $0.65=0.65 \times 100 \%=65 \%$
3- Finding a percentage
Example1: Find $10 \%$ of $80=\frac{10}{100} \times 80=\frac{800}{100}=8$
Example2: Find $5 \%$ of $60=\frac{5}{100} \times 60=\frac{300}{100}=3$
4- Changing into percentages

Example 1: 30 out of $50=\frac{30}{50} \times 100 \%=\frac{3000}{50}=60 \%$
Example 2: 45 out of $60=\frac{45}{60} \times 100 \%=\frac{4500}{60}=75 \%$
5- Increasing /decreasing by a percentage amount.
Example1: Increase 30 by $10 \%$

$$
10 \% \text { of } 30=3
$$

Therefore increasing 30 by $10 \%$ gives $30+3=33$

Example2: Decrease 40 by 5\%.

$$
5 \% \text { of } 40=2
$$

Therefore decreasing 40 by $5 \%$ gives $40-2=38$

### 1.15 Ratios:

ARATIO is the relation existing between two or more quantities of the same kind and there is very much a direct link between fractions and ratios.
Example: $2: 6=\frac{2}{6}, 5: 15=\frac{5}{15}, 8: 24=\frac{8}{24}$
All these fractions cancel down to $\frac{1}{3}$. Hence the ratio in its simple form would be 1:3.

Notes:
i) Since ratio is a comparison of sizes (i.e. Lengths, masses, amounts of money etc.) no units are involved. For example we have two lines as shown in the figure (1.12)
Line A


Line B

Fig.(1.12)
The ratio of line A to line B is $1 \mathrm{~cm}: 3 \mathrm{~cm}$
i.e. 1:3 ( a comparison of the measurements - units not required)
ii) We must be certain that units are of the same type before we can compare them in ratio form.
Example: What is the ratio of 1 m to 1 km ?
The ratio would be $1 \mathrm{~m}: 1 \mathrm{~km}$, which is, of course,
$1 \mathrm{~m}: 1000 \mathrm{~m}$, i.e. $1: 1000$
ii) Applications of ratio:

Provided that we are given enough information in the form of ratios and figures we can use the information to solve the problem.
Example 1: Divide 100 dinar in the ratio 1:4
Altogether five shares are required (i.e. $1+4$ )
Each share is worth $\frac{100}{5}=20$ dinar
Hence 1 share is 20 and four shares will be $4 \times 20=80$
Thus $£ 100$ divided in the ratio $1: 4$ gives $£ 20: £ 80$
Example 2: In a class of 30 pupils the ratio of the number of boys to the number of girls is $7: 8$ How many girls are there?

Boys: Girls
7 : 8
We need altogether 15 shares (i.e. $7+8$ )
One share will be $\frac{30}{15}=2$ pupils
Hence since the girls account for eight shares, there must be
$8 \times 2=16$ girls in the class (there would of course be 14 boys)
Example 3: The ratio of the length of a room to its width is $4: 3$
How wide is a room that is 10 m long?
Length : Width

$$
4: 3
$$

Hence four shares $=10 \mathrm{~m}$ and so 1 share $=\frac{10}{4}=2 \frac{1}{2} \mathrm{~m}$
Now width is three shares i.e. $3 \times 2 \frac{1}{2} \mathrm{~m}=7 \frac{1}{2} \mathrm{~m}$

### 1.16 Proportion:

A proportion is a statement of equality between two ratios as:

$$
1: 3=2: 6=10: 30 \text { i.e. } \frac{2}{6}=\frac{10}{30}=\frac{1}{3}
$$

## Types of proportion:

## 1- Direct proportion:

Two quantities are in DIRECT PROPORTION if an increase (or decrease) in one quantity is matched by an increase (or decrease) in the same ratio in
the second quantity. The type of problem can be seen in the following example.
Example: 8 blocks of chocolate cost 96 p. How much will
( a ) 5 blocks of chocolate cost ?
(b) 13 blocks of chocolate cost?

The problem is best solved by finding the cost of 1 chocolate block i.e.

8 blocks cost $96 \mathrm{p}: .1$ block costs $\frac{96}{8} \mathrm{p}=12 \mathrm{p}$
Hence (a) 5 blocks cost $5 \times 12 \mathrm{p}=60 \mathrm{p}$
And (b) 13 blocks cost $13 \times 12 \mathrm{p}=156 \mathrm{p}=£ 1.56$
This can be interpreted as :

| 8 <br> cost <br> 5 <br> cost <br> $\frac{8}{5}=\frac{96}{x}$$\xrightarrow{96}$ |  |
| :--- | :--- |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  | $8 x=\frac{96}{8} \times 5=96 \times 5$ |

## 2-Inverse proportion:

If an increase (or decrease) in one quantity produces a decrease (or increase) in a second quantity in the same ratio, then the two quantities are said to be in INVERSE PROPORTION.

## Example:

For instance, if it takes 8 days for 2 men to do a job, how long would it take 4 men to do the same job?
When we study the problem closely, it is obvious that if more men do the same job, then the job is completed in a much faster time - in this case 4 men will do the job in half the time, i.e., 4 days.

Remember to check your result to see if it is reasonable; in the example above, it is reasonable to assume that 4 men will do the job twice as quickly as 2 men, and on the other hand 1 man would take twice as long to complete the job, i.e. 16 days.

### 1.17 Directed numbers:

Directed numbers are the positive (+ ve) and negative (-ve) real numbers.

We indicated before that the set of real numbers (R) is made up of positive numbers, negative numbers, and zero, which is neither (+ve) nor (-ve).

## Notes:

1) Each (+ ve) number has associated with it one and only one (-ve ) number and each ( - ve ) number has associated with it one and only one ( + ve ) number.
2) If we denote the set of real numbers by ( R ), the set of ( + ve ) numbers by ( $\mathrm{R}^{+}$), The set of negative numbers by ( $\mathrm{R}^{-}$), and the set that has zero as the only element by $\{0\}$, then
$R=R^{+} U R^{-} U\{0\}$, this indicates that zero (0) is neither + ve nor -ve it is a neutral number.
3) Three are several ways of denoting directed numbers such as:
(i) $-{ }^{+} 3,-2$ or (ii)- $+3,-2$
4) The set of real numbers ( $R$ ) can be represented geometrically (graphically) by the set of all points on a straight line called a real numbers line or the real line, so that each point on it is assigned to a real number, this means that there is a one-to-one correspondence between the real numbers and the points on the real line, the number assigned to a point is called the coordinate of the point the point assigned to a number is called its graph.
5) The real line extends to infinity ( $\infty$ ) from both sides, as shown in the figure (1.13).


Fig.(1.13)
6) Directed numbers are marked off uniformly on the real line by choosing an arbitrarily point representing zero called the origin or reference point, Then the $(+\mathrm{ve})$ real numbers are marked off on the right of zero and the (-ve) numbers are marked off on the left of zero.
7) The marking system of the real line is according to a mathematical convention which shows that the (+ ve) numbers to the right of (0) get
bigger and bigger as we go a long the number line to the right, and the (- ve) numbers to the left of $(0)$ get smaller and smaller as we go along the number line to the left.

## Example:

$$
\begin{array}{llll}
8>5 \\
3<7
\end{array}, \quad, \quad+2>-6, \quad-3>-9
$$

8) The directed numbers may be indicated by directed moves on the real line. as shown in the figure (1.14).


Fig.(1.14)

### 1.18 Operation on directed numbers:

The four operations on directed numbers (addition - subtraction multiplication - and division) are interpreted by the use a series of directed moves on a number line as follows:

## 1- Addition and subtraction:

These two operations are linked together and may be interpreted as a series of directed moves on a number line such as in the figures(1.15-a, b, c, d):

## Examples:



Fig.(15-a)
ii) $-4-3=-4+(-3)=-7$


Fig.(15-b)
iii) $2-3=2+(-3)=-1$ or $-3+2=2-3=-1$


Fig.(15-c)
iv) $-4+3=4-(-3)$

$$
=3+(-4)=-1
$$



Fig.(15-d)

## Notes:

1- According to the last illustrations we can abstract the following sign rules associated with addition and subtraction of directed numbers:
i) $(+$ ve $)+(+$ ve $)=+$ ve result $(-$ ve $)+(-$ ve $)=-$ ve result $\quad\}$ by addition of the two
ii) $(+\mathrm{ve})+(-\mathrm{ve})=+\mathrm{ve}$ or -ve numbers.
$(-\mathrm{ve})+(+\mathrm{ve})=+\mathrm{ve}$ or -ve as the sign of the
iii) $a+(-b)=a-b$
iv) $a-(-b)=a+b$
v) $(a+b)-b=a$
vi) $-(a+b)=-a-b$

## Examples:

Perform each of the following operations:
i) $(+6)+(+3)=6+3=9$
ii) $(-6)+(-3)=-6-3=-9$
iii) $(-4)-(-5)=-4+5=1$
iv) $(+1)-(+5)=1-5=-4$
v) $(+7)+(-3)=7-3=4$
vi) $(-8)+(+2)=-8+2=-6$
vii) $(5+3)-3=5+3-3=5$
viii) $-(6+2)=-6+(-2)=-6-2=-8$
1)

The sum and difference of directed numbers can be illustrated by a formal way using the additive inverse property as follows:

## Examples:

Perform each of the following operations:
i)

$$
8+(-3)=(5+3)+(-3)
$$

$$
=5+(3-3)
$$

$$
=5+0=5
$$

$\therefore 8+(-3)=8-3=5$
ii)

$$
-15+7=(-8-7)+7
$$

$$
=-8+(-7+7)
$$

$$
=-8+0=-8
$$

$$
\therefore-15+7=7-15=-8
$$

iii) $-9+(-6)=-9-6$

$$
\begin{gathered}
-(15-6)-6=-15+6-6 \\
=-15 \\
\therefore-9-6=-15
\end{gathered}
$$

## 2-Multiplication and division:

Using the ideas of addition and multiplication of real numbers and their properties indicated before we can make the following statements concerning the directed numbers:
i) $a \cdot(-b)=-a \cdot b=-(a b)$
ii) $a(b-c)=a b-a c$
iii) $a \div b=\frac{a}{b}=a \cdot \frac{1}{b}$ if $b \neq 0$
iv) $a \div(-b)=-a \div b=\frac{a}{-b}=\frac{-a}{b}=-\frac{a}{b}$
v) Multiplication and division of directed numbers follow the same pattern of signs.

- $\quad(+$ ve $) \times, \div(+$ ve $)=+$ ve result
- $(-$ ve $) \times, \div(-$ ve $)=+$ ve result
- $\quad(+$ ve $) \times, \div(-$ ve $)=-$ ve result
- $(-$ ve $) \times, \div(+$ ve $)=-$ ve result

Examples: Perform each of the following operations:
i) $3 \times 4=4+4+4=12$
ii) $3 \times(-4)=-(3 \times 4)=-12$
iii) $-3 \times 4=-(3 \times 4)=-12$
iv) $-3 \times-4=-(3 \times-4)$

$$
\begin{aligned}
& =-(-(3 \times 4)) \\
& =-(-12)=12
\end{aligned}
$$

v) $6 \div 3=\frac{6}{3}=6 \times \frac{1}{3}=2$
vi) $6 \div(-2)=\frac{6}{-2}=6 \times \frac{1}{-2}=-3$
vii) $-6 \div 2=\frac{-6}{2}=-6 \times \frac{1}{2}=-3$
viii) $-6 \div(-2)=\frac{-6}{-2}=-6 \times \frac{1}{-2}=3$

## Unit. 2: Geometric concepts.

2.1 - Geometric figures.
2.2 - Polygons.
2.3 - Circle.
2.4 - Solid figures.

## Unit. 2: Geometric concepts

## Introduction.

Elementary geometry is the part of mathematics which deals with the study of relationships between Geometric figures in space.

It started with an in defined concept called a point which was considered as a foundation stone in the building of this subject, if we join two or more points, we get a geometric figure which is a shape drawn in space it could be in one dimension such as straight line or in two dimension such as rectangle or in three dimensions such as cube.

This means that the study of geometry deals with learning about geometric shapes and structures and how to analyze their characteristics and relationships.

In this section we will introduce some elementary geometric concepts.

### 2.1 Geometric figures:

## 1- The point

The point is an in defined concept it has apposition but it has no magnitude that is no size, and we represent it geometrically by a dot as: A

## 2- Straight line:

The straight line is a geometric figure formed by joining two points in a way that the distance between any three points equals the sum of the distances between each two successive points as shown in the figure (2.1).


Fig.(2.1)
i.e. $\quad d=d_{1}+d_{2}$ for any three points

## Note:

The straight line extends to infinity from both sides so it does not have ends so that we cannot measure its length because its infinite.

## 3- line segment:

The line segment is a part of a straight line.

## Note:

We can measure the length of the line segment because it is finite (see the figure (2.2).


Fig.(2.2)

## 4- Angle:

a) The angle is a geometric figure formed by meeting two half straight lines (or line segments) in a point as shown in the figure(2.3).


Fig.(2.3)

It can be defined also as follows: The angle is a geometric figure formed by rotating a line segment about a point from an initial position to a terminal position as shown in the figure(2.3).


Fig.(2.4)
Note:
b) An angle is called positive if the direction of rotation is counter clockwise and negative if it is clockwise, as shown in the figure(2.5).


Fig.(2.5)
c) The measure (size) of the angle is the amount of rotation which the side of the angle rotated from its initial position to its terminal poison, the unit of
measurement is called degree, there are $\left(360^{\circ}\right)$ degrees in a complete revolution the degree is divided into (60') minutes and the minute is divided into (60 ") seconds.
d) The angle is named according to its size, there are five types of angles , as shown in the figure(2.6-a, b, c, d,e,f).

a)- Acute angle

$$
\mathbf{0}^{\circ}<\mathrm{x}<90^{\circ}
$$


c)- Obtuse angle

$$
90^{\circ}<\mathrm{z}<180^{\circ}
$$


b)- Right angle

$$
Y=90^{\circ}
$$


d)- Straight angle
$\mathrm{W}=\mathbf{1 8 0}^{\circ}$

e) $\mathbf{1 + 2}=\mathbf{9 0}$ called e)-complementary



Supplementary angles

Fig.(2.6)
e) When two lines cross each other at a point we get vertically opposite angles which are equal, as shown in the figure (2.7).
i.e.
$1=2$
$3=4$


Fig.(2.7)
f) If a straight-line cut across two parallel lines w get equal angles, as shown in the figure (2.8).

1
a) $\mathbf{1 = 2}$ corresponding angles

b) $\mathbf{1 = 2}$ Alternate angles

Fig.(2.8)

## 5- Plane:

The plane is a flat surface in which all points of a straight line joining any two points of this surface will belong to the surface as shown in the figure (2.9).

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    أ. عبير خليل صليبي
```



Fig.(2.9)
i.e. if $\mathbf{A}, \mathbf{B} \in \mathbf{P}$
$A, B \in L\}$
$\mathbf{L} \subset \mathbf{P}$

## Note:

The plane extends to infinity from all sides, so we cannot measure its dimensions (length, breadth).

## 6- Plane figure:

The plane figure is a Geometric figure drawn in the plane, bounded by a closed curve or a set of connected line segments.

## Note:

a)

The plane figure is a two-dimensional figure.
b) The plane figure can be rectilinear or non rectilinear according to boundaries as shown in the figure ( $2.10-\mathrm{a}, \mathrm{b}$ ):


Fig.(2.10- a) - Rectilinear plane figures


Fig.(2.10-b)- Non rectilinear plane figures

### 2.2 Polygons:

The polygon is a plane rectilinear figure bounded by a set of closed line segment.

## Note:

The polygon is called by the number of its sides( see the figures(2.11)), such as:
a) Three sides are called a triangle.


Fig.(2.11-a)
b) Four sides are called quadrilateral


Fig.(2.11-b)
c) Five sides are called pentagon
d) $6=$ hexagon


Fig.(2.11-c)


Fig.(2.11-d)


Fig.(2.11-e)

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f) $8=$ octagon
g) $9=$ Nonagon

h) $10=$ decagon

i) The perimeter (circumference) of the polygon is the distance all around its edges (sides).
j) The area of the polygon is the amount of surface bounded by its circumference.
k) The vertices of the polygon are the points where its sides meet.
l) The diagonal of the polygon is a line segment joining two opposite vertices.see the figure (2.12)


Fig.(2.12)
m) The regular polygon is a polygon whose all sides and all angles are equal such as, sequare, hexagon,.... . see the figure (2.13)


Fig.(2.13)

## Triangle:

The triangle is a polygon with three sides see the figure (2.14)


Fig.(2.14)

## Notes:

a) The triangle has three angles, three sides, three vertices.
b) The triangle is named by its sides or its angles as shown in the figure (2.15-a,..,f):

a)-Scalene triangle

No sides or angles equal

c)- Equilateral triangle all sides and angles are equal

e)- Acute angled triangle all angles acute

b)- Isosceles triangle

Two sides and two angles equal

d)- Right-angle triangle one angle equal $90^{\circ}$

f)-Obtuse angled triangle one angle obtuse

Fig.(2.15)
c) The height (altitude) of the triangle is a perpendicular line segment from a vertex on the opposite side.
d) The base is the side which is perpendicular to the height.
e) The median of the triangle is a line segment joining vertex and the midpoint of the opposite side. As shown in the figure (2.16).


Fig.(2.16)
f) The angle bisector of the triangle is the line segment which bisects an interior angle and extends from the vertex of that angle to the opposite side(see the figure (2.17).


Fig.(2.17)

## Quadrilateral:

The quadrilateral is a polygon with four sides as shown in the figure (2.18).



Fig.(2.18)

## Note:

Any polygon with four sides is called a quadrilateral, we will introduce here the most famous shapes those which have been treated in the school mathematics, and we will list them in such away according to an order from
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a whole shape which is in this case the trapezium to its parts, i.e., from a general shape to its special cases as follows:

## Trapezium:

The trapezium is a quadrilateral with two opposite sides are parallel as shown in the figure (2.19).


Fig.(2.19)

## Notes:

a) The two parallel sides of the trapezium are called its bases. The line segment joining the mid points of the other two sides is called the middle bas.
The meddle base is parallel to the other two parallel bases and equals half of the sum of their lengths.
i.e., middle base $=\frac{1}{2}$ ( small base + big base $)$
b) The isosceles trapezium is a trapezium has the non-parallel sides equal (see the figure (2.20))


Fig.(2.20)
c) The right trapezium is a trapezium which one of its angles $\left(90^{\circ}\right)$. (see the figure (2.21))


Fig.(2.21)

## Parallelogram:

The parallelogram is a trapezium whose parallel opposite sides are equal.

- Properties of the parallelogram:
a) opposite sides are parallel and equal.
b) opposite angles are equal.
c) Diagonals bisect each other.
d) Each diagonal divides it into two congruent triangles.
e) The height of the parallelogram is the perpendicular line segment from one of its vertices on one of its sides, so the parallelogram has two heights and two bases. (see the figure (2.22))


Fig.(2.22)

## Rectangle:

The rectangle is a parallelogram with one of its angles is $\left(90^{\circ}\right)$.

- properties of the rectangle:

Since the rectangle is a special case of parallelogram, then it has all of its properties in addition to:
a) All angles are ( $90^{\circ}$ right).
b) Diagonals are equal. (see the figure (2.23))


Fig.(2.23)
c) The smallest side in the rectangle is called the height or breadth or width, and the longest side is called the length or base, the height and base are called the dimensions of the shape.

## Square:

The square is a rectangle whose all sides are equal.

- Properties of the square:

Since the square is a special case of rectangle, then it has all of its properties in addition to:
a) All sides are equal.
b) Diagonals are perpendicular. (see the figure (2.24)).


Fig.(2.24)

## v) Rhombus:

The rhombus is a parallelogram with all sides equal. (see the figure (2.25)).


Fig.(2.25)

- Properties of the rhombus:

We would like to show that the rhombus is outside the last order of familiar shapes it is a special case of the parallelogram so it has all its properties in addition to:
a)

All sides are equal.
b)

Diagonals are perpendicular.
c)

The square is a special case of the rhombus.

## Note:

The previous description of the familiar quadrilaterals shows the relationship between them, it shows that the square is a special case of the rectangle which is a special case of the parallelogram and the trapezium and the trapezium is a special case of the quadrilateral. The rhombus is an exception it is a special case of the parallelogram and the square is a special case of the rhombus.

### 2.3.Circle:

There are different definitions to the circle all give the same information's about this figure, we will suggest some of them in the following.

- The circle is a plane figure bounded by a closed curve such that each point on it is at a fixed distance from a fixed point in the same plane.
- The circle is the locus of a point moving in the plane such that it is at a fixed distance from a fixed point in the same plane.
- The circle is a set of points in the same plane which are at a fixed distance from a fixed point in the same plane.


## From the figure (2.26) we can note that:



Fig.(2.26)
a) The fixed point is called the center.
b) The fixed distance is called the radius.
c) The chord of the circle is a line segment joining two point on the circumference.
d) The diameter is a chord passing through the center.
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e) The radius is half of the diameter.
f) The secant is a line passing through two point on the circumference.
g) The tangent is a line touches the circle at a point on the circumference.
h) The central angle is an angle between two radii.
i) The sector is a part of the area of a circle cut off by two radii.
j) The segment is a part of the circle cut off by a chord.(see the figure (2.27-a,b))


Fig.(2.27-a)


Fig.(2.27-b)

### 2.4 Solid figure:

The solid figure is a geometric figure drawn in space and bounded by a closed surface.

## Notes:

a) The solid figure is a three-dimensional figure i.e., it has three dimensions (length, breadth, and height) as shown in the figure (2.28).
b) The solid figure called rectilinear if all its edges are straight line segments, otherwise it is called non rectilinear.
c) The rectilinear solid is called also a polyfaces, which faces are planes. We shall look at the names and properties of a few simple solids.
i) RECTILINEAR solids - ones in which all the edges are straight lines.


Fig.(2.28)
$\boldsymbol{A B C D}$ is one of the FACES (surfaces) and forms parts of a (flat surface) PLANE.
This solid has 6 FACES (F), 8 CORNERS (C) and 12 EDGES (E) and there is an important statement about rectilinear solid called EULER 'S THEOREM.
It says $\mathbf{F}+\mathbf{C}-\mathbf{E}=\mathbf{2}$.

As shown in the figure (2.29) we can see:.
(a) CUBE
(b) TRIANGULAR PRISM
(c) HEXAGONAL PRISM
(d) PYRAMID
(e) PYRAMID or TETRAHEDRON
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ii) Non - RECTILINEAR

From the figure (2.30) we can see some bodies which are nonrectilinear such as:


## CYLINDER



CONE


THE FRUSTUM
OF ACONE


SPHERE

Fig.(2.30)

## Unit. 3. Algebraic concepts

3.1 - Sets.
3.2 - Symbolic representations.
3.3 - The algebraic method of solving equations.
3.3.1 - Linear equations in one variable.
3.3.2 - Quadratic equations.
3.3.3 - Simultaneous equations.
3.4 - The algebraic method of solving in equalities.
3.4.1 - Linear in equalities in one equalities.
3.4.2 - Quadratic in equalities.
3.4.3 - Simultaneous in equalities.

## Unit. 3: Algebraic concepts

## Introduction

The word " algebra " had its origin in the Arabic word " aljabr " which meant " reduction of parts to the whole " and was employed for example, in the context of bone - setting. it came in to the westren world round a bout (1200) as a part of the title of a book " Hisab aljabr w al muqabalah " written by " AL - khowarizmi ".
In which he gave an account of " Hindu and Babylonian method for the solution of equations. The contemporary translation in mathematical terms to the title of would be " The science of transposition and cancellation " the book was intended to be used by astronomers, and so contained many prescriptions for the solution of equations. Thus " algebra " to most educated people from that time on, meant the solving of equations, and the development of formulae.
Before 1960 there was widespread agreement about the nature of school algebra, it was regarded as a generalization and extension of arithmetic. The above view point is a classical one which adapted what is called the traditional algebra, which went on to include the art and theory of manipulating symbols, it is in much more recent times that we have seen the emergence of " modern algebra " which is concerned with the study of structure which is interred school algebra only in the few last years, such topics are: abstract algebra, linear algebra, boolen algebra, matrix algebra, the algebra of sets and functions, ...etc.

### 3.1 Sets:

## 1-Set definition.

A set is a well-defined collection of distinguishable objects.

## Notes:

a) The objects in the set are called elements or members.
b) Well - defined means that the definition of the set must be by a clear rule which according to it we can decide whether or not a particular object is a member of the set.
c) Distinguishable means that the elements of the same set are different from each other, so elements do not repeat in the same set.
d) The order in which the elements are listed is unimportant, mathematically.

## 2-Equal sets:

Two sets are equal if they contain the same elements.

$$
\text { i.e. } \quad \mathrm{A}=\mathrm{B} \quad \longleftrightarrow x \in \mathrm{~A} \quad \longleftrightarrow x \in \mathrm{~B}
$$

## 3-Finite and infinite sets:

A set is finite if it consists of a specific number of elements.
Otherwise, it is infinite.

## 4-Null set:

The null (empty) set is a set which contains no elements, it is denoted by the letter $\varnothing$.

## 5-Subset:

The set $A$ is called a subset of $B$ if every element of $A$ is also an element of B. see the figure (3.1)


Fig.(3.1)

## 6-Power set:

The power set of $A$ is the set of all its subsets.

## 7-Union:

The Union of two sets $A$ and $B$ is the set of all elements which belong to $A$ or B , as shown in the figure $(3.2-\mathrm{a})$

$$
\text { i.e. } \quad A \cup B=\{x: x \in A \quad \text { or } x \in B\}
$$

## 8-Intersection:

The intersection of two sets A and B is the set of all elements which belong to both A and B . shown in the figure (3.2-b)
i.e. $A \cap B=\{x: x \in A$ and $x \in B\}$

## Note:

Two sets A and B are called disjoint if $A \cap B=\varnothing$ as shown in the figure (3.2-c)


Fig.(3.2)

## 9-Universal set:

The universal set $(\mathrm{U})$ is a set which contains all set under investigation.

## 10- Complement:

The complement of the set $\mathrm{A}\left(A^{\prime}\right)$ is the difference between U and A . i.e. $\mathrm{A}^{\prime}=\mathrm{U}-\mathrm{A}$, as shown in the figure (3.3)


Fig.(3.3)

## 11- Difference between two sets:

The difference between two sets A and B is the set of all elements which belong to A and does not belong to B .
i.e., $A-B=\{x: x \in A$ and $x \notin B\}$, as shown in the figure (3.4)


Fig.(3.4)

### 3.2 Symbolic representations:

## 1-Variable:

The variable is a symbol used to denote any element of a given set.

## Notes:

a) Variables are represented by symbols (letters) such as $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.
b) The variable sometimes is called an unknown.

## 2-Constant:

The constant is a symbol stands for one fixed number.
e.g. $\boldsymbol{x}=\mathbf{3}, \boldsymbol{y}=-2$.

## 3- Algebraic term:

The algebraic term is an algebraic quantity in the form of a constant number, or a number times a variable.

$$
\text { e.g. } 3, \frac{2}{3}, 4 x, 3 x^{2}
$$

## Notes:

a) The coefficient of the algebraic term is the constant part in the term.
e.g. The term $4 \boldsymbol{x}$ its coefficient is 4 .

The term $-3 y$ its coefficient is -3 .
b) The degree of the algebraic term is the sum of the powers of its variables.
e.g. The term $x^{2} \boldsymbol{y}^{3}$ its degree is 5 .
c) The like algebraic terms are terms with same variables having same degree
e.g. The terms $2 x, 5 x$ are like terms.
e.g. The terms $2 x y,-2 x y$ are like terms.
e.g. The terms $x^{2} y, y x^{2}$ are like terms.
e.g. The terms $x^{3}, y^{3}$ are un like terms.

## 4- Algebraic expression:

The algebraic expression is a set of algebraic terms connected by addition, subtraction operations.
e.g. $3 x^{2}-4 x+1, y^{3}-2 x-3 y$

## Notes:

a) The algebraic expression is called by the number of its terms.
b) The degree of the algebraic expression is the highest degree of its terms.
c) The value of the algebraic expression is the number which this expression represents when we substitute with the values of the variables in it e.g., Evaluate $x y+\frac{z^{2}}{2}$

$$
\text { When } x=2, y=-3, z=4
$$

Substituting:

$$
\begin{aligned}
x y+\frac{z^{2}}{2} & =2(-3)+\frac{4^{2}}{2} \\
& =-6+\frac{16}{2} \\
& =-6+8 \\
& =2
\end{aligned}
$$

## 5-Mathematical sentence:

The Mathematical sentence is a statement which gives mathematical information.

## Notes:

There are three types of sentences:
a) A false sentence which gives false mathematical information.

$$
\text { e.g. } \mathbf{7 > 1 0}, \mathbf{1 2} \div \mathbf{6}=\mathbf{4}
$$

b) A true sentence which gives true mathematical information.

$$
\text { e.g. } 3+2=5,4 \times 3=12
$$

c) An open sentence: which gives false or true mathematical information according to the values of its variables.
e.g. $\boldsymbol{x}+\mathbf{3}=\mathbf{5}$ is true when $\boldsymbol{x}=\mathbf{2}$ and false otherwise.

## 6- Equation:

The equation is an open sentence has two equal sides each in the form of an algebraic expression connected by an equals sign.

$$
\text { e.g. } \quad 3 x+2=x-3
$$

## Notes:

a) The solution of the equation is the value of the variable which makes it A true sentence.
b) The solution set of the equation is a set which its elements are the solutions of this equation.
c) Solving the equation means finding its solutions.
d) Two equations are equivalent if they have the same solution set.

## 7- Formula:

The formula is an equation which has arisen from mathematical application.
e.g., $\quad A=\boldsymbol{\pi} \boldsymbol{r}^{\mathbf{2}}$ ( area of the circle).

## 8- Identity:

The identity is a true mathematical sentence which has two equal sides in the form of expression which are equal for all values of the variables.
e.g. $(x-y)(x+y)=x^{2}-y^{2}$

## 9- Inequality:

The inequality is an open sentence has two in equal sides each in the form of an algebraic expression.

$$
\text { e.g. } x-3>2 x-1
$$

### 3.3 The algebraic method of solving equations:

It is mentioned in the previous section of this unit that the equation is an open sentence having two equal sides each in the form of an algebraic expression, connected by an equals sign.
Such as $2 \boldsymbol{x}-\mathbf{3}=\mathbf{5}$ and $\boldsymbol{x}^{2}+\mathbf{3 x}-\mathbf{1}=\mathbf{0}$.

The name of the equation differs according to the number of variables unknowns and its degree.
Such as: $a x+b=0, a \neq 0$ is a first degree in one variable. $a x^{2}+b x+c=0, a \neq 0$ is a second degree in one variable. $a x+b y+c=0, a \neq 0$ is a first degree in two variables. $a x^{2}+b x+c y+d=0$ is a second degree in two variables. it is called an equation since it contains the verb equals ( $=$ ). The symbols $x$ and $y$ are called variables or unknowns. To solve the equation is to find those values of the variables ( $\boldsymbol{x}$ and $\boldsymbol{y}$ ) which when they replace them turn the open sentence into a true statement, i.e. they satisfy it. There are different methods of solving we will discuss in this section the algebraic method which uses algebraic operations based upon the properties of real numbers and equations as follows:

### 3.3.1 Linear equation in one variable:

Is an equation in the form $\boldsymbol{a x}+\boldsymbol{b}=\mathbf{0}, \mathbf{a} \neq \mathbf{0}$ which is also called a first degree equation in one variable (unknown).
To solve this type of equations a number of methods are suggested by different. School mathematics text books, they all come to one method called the isolating method, the objective here is to reduce the left hand side to the variable $(\times)$ alone, to achieve this we separate the variables from the numbers and isolating them the symbols are put in the left hand side and numbers in the right hand side by a successive transformations to the equation keeping the sides equal to one another following a series of an algebraic operations using the concepts and properties of real numbers such as: binary operation of $(+$ and $\times$ ), identify identity elements with respect to these operations $(+, \times$ ) which are ( 0 and 1 ) inverse elements with respect to the operations inverses, associative and commutative lows.
From these laws and the equality properties:

$$
a=b \rightarrow a \pm c=b \pm c \quad \text { and } \quad a=b \rightarrow a c=b c \quad \text { and } \quad \frac{a}{c}=\frac{b}{c}
$$

We deduce the golden rule of equations, which says that you may add or subtract anything on both sides of an equation, and that you may multiply or divide both sides of an equation by anything except zero, which is used in solving equations.

## Example:

Solve the equation $2 \boldsymbol{x + 3}=\mathbf{1 1}$.

## Solution:

Subtract 3 from both sides of the equation $\mathbf{2 x + 3}=\mathbf{1 1}$ i.e. add the additive inverse of $3(-3)$ to both sides.

$$
\left.\begin{array}{rl}
\therefore 2 x+3=11 & \longrightarrow 2 x+3-3=11-3 \\
& \text { Or } 2 x+3+(-3)=11+(-3)
\end{array}\right\} \Rightarrow 2 x=8
$$

$\therefore$ Divide each side by or multiply each side by the multiplicative inverse of $2\left(\right.$ which is $\frac{1}{2}$ )
$\therefore \quad \frac{2 x}{2}=\frac{8}{2} \quad \Longrightarrow \quad x=4$
Or $\frac{1}{2} \cdot 2 x=\frac{1}{2} .8 \quad \Longrightarrow \quad x=4$
$\therefore$ The solution of the equation is 4 and the solution set is $S=\{4\}$.

- We would like to indicate here that the above method suggests a short and quick method for solving an equation which is known as transferring the terms from one side of an equation to the other with changing the sign of the term transferred, we will point out in the following example that this operation is as a result of adding the additive inverse of the term transferred from one side to the other to both sides of the equation. and we will indicate also that this method is applied to all types of equations as well as inequalities.


## Example:

Solve the equation $3 x-2=5 x+6$.
Solution: we will follow the previous steps then we deduce the note we mentioned.
$3 x-2=5 x+6$
$3 x-2+2=5 x+6+2$
$3 x=5 x+6+2$
$3 x-5 x=5 x+6+2-5 x$
$3 x-5 x=6+2$
If we pass from step (i) to step (v) directly we notice that the terms ( -2 ) and ( 5 x ) are transferred from their sides to the other side with change of their signs.

$$
\therefore 3 x-5 x=6+2 \longrightarrow-2 x=8
$$

Divide each side by -2 or multiply by $-\frac{1}{2}$
$\therefore \frac{-x}{-2}=\frac{8}{-2} \Longrightarrow x=-4$
$\therefore$ the solution of the equation is 4 and the solution set is $S=\{4\}$.

## Example:

Solve the equation $2 x-5=4-x$.
Solution: we will use the short method as:
$2 x-5=4-x \Longrightarrow 2 x+x=4+5$
$\therefore 3 x=9$, divide by 3 to get $\quad x=3 \quad \therefore S=\{3\}$.
Note: There is another method used to solve simple linear equations called inverse operations method which is done by drawing a flow-chart as shown in the figure (3.5). This flow chart illustrates how the left-hand sides is built up then reverse the flow-chart as shown in the following example: solve the equation
$2 x+3=11$.

R.H.S
$\therefore x=4$
$S=\{4\}$.
Fig.(3.5)

### 3.3.2 Quadratic equation:

It's an equation in the form:
$\boldsymbol{A} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}=\mathbf{0}$, where $\boldsymbol{a}, \boldsymbol{b}$ and $c$ are real numbers, $\boldsymbol{a} \neq \mathbf{0}$ and $x$ is the variable (unknown). It is also called a second - degree equation in one variable (unknown), thus it is called (quadratic). A quadratic equation is called incomplete if either its constant term is zero or its first- degree term is missing i.e. $\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}=\mathbf{0}$ and $\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{c}=\mathbf{0}$ are both incomplete equations.

The main algebraic solving methods of a quadratic equation are factoring method and quadratic formula method in which the property of real numbers, $\boldsymbol{a b}=\mathbf{0} \Longleftrightarrow \boldsymbol{a}=\mathbf{0}$ or $\boldsymbol{b}=\mathbf{0}$ or both is used.
I - Examples on factoring method.
Solve each of the following equations.

1) $x^{2}-8 x+15=0$, factoring the left-hand side (L.H.S)
$(x-3)(x-5)=0$, then

$$
x-3=0 \quad \Longrightarrow x=3, x-5=0 \quad \Longrightarrow x=5
$$

$\therefore$ the solution set is $S=\{3,5\}$.
2) $3 x^{2}-x-14=0$, factoring the (L. H. S)

$$
\begin{aligned}
& (3 x-7)(x+2)=0, \text { then } \\
& 3 x-7=0 \quad \Longrightarrow x=\frac{7}{3}, x+2=0 \Longrightarrow x=-2
\end{aligned}
$$

$\therefore$ the solution set is $S=\left\{-2, \frac{7}{3}\right\}$.
3) $x^{2}-16=0$, factoring the (L. H. S)

$$
(x-4)(x+4)=0 \quad \text { then } \quad x=4 \text { or } x=-4
$$

$$
\therefore S=\{-4,4\} .
$$

4) $5 x^{2}-6 x=0$, factoring the (L. H. S) $x(5 x-6)=0$, then $x=0$ or $x=\frac{6}{5}$

$$
\therefore S=\left\{0, \frac{6}{5}\right\}
$$

$\Pi$ - Solving the general quadratic equation $\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b x}+\boldsymbol{c}=\mathbf{0}$,
( $a, b$ and $c$ are reals , $a \neq 0$ ) by completing the square enables us to derive a formula for finding the solutions to a quadratic equation in terms of the coefficients $a, b$ and $c$, solving this equation by completing the square involves the following steps:
i) Put the constant term on the right-hand side: (R.H.S).
$a x^{2}+b x=-c$.
ii) Divide through by the coefficient of $\boldsymbol{x}^{2}$.
$x^{2}+\frac{b}{a} x=-\frac{c}{a}$.
iii) add $\left(\frac{\mathrm{b}}{2 \mathrm{a}}\right)^{2}$ (the square of $\frac{1}{2}$ of the coeffient of $x$ ) to both sides.

$$
\begin{aligned}
x^{2}+\frac{b}{a} x+\left(\frac{b}{2 a}\right)^{2}=-\frac{c}{a}+\left(\frac{b}{2 a}\right)^{2} \\
x^{2}+\frac{b}{a} x+\frac{b^{2}}{4 a^{2}}=\frac{b^{2}}{4 a^{2}}-\frac{c}{a}
\end{aligned}
$$

$$
\left(x+\frac{b}{2 a}\right)^{2}=\frac{b^{2}-4 a c}{4 a^{2}} .
$$

iv) taking square root of both sides.
$x+\frac{b}{2 a}= \pm \sqrt{\frac{b^{2}-4 a c}{4 a^{2}}}$
$\therefore x=-\frac{b}{2 a} \pm \frac{\sqrt{b^{2}-4 a c}}{2 a}$
$\therefore x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$
This is called the quadratic formula or the general law of solving a quadratic equation.

## Notes:

1) The linear equation $\boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}=\mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$ has only one solution (root) which is $\boldsymbol{x}=-\frac{\boldsymbol{b}}{\boldsymbol{a}}$.
2) In $\boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}=\mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$ if $\boldsymbol{b}=\mathbf{0}$ then $a x=0$ then $x=0$.
3) In $\boldsymbol{a x}+\boldsymbol{b}=\mathbf{0}$ if we allow $\boldsymbol{a}=\mathbf{0}$ then it becomes $b=a$, which is not an equation, thus $\boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}=\mathbf{0}$, a must not equals 0 .
4) A quadratic equation $\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}=\mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$ which has rational roots can be solved by factoring, but equations with irrational roots can be solved only by completing the square or by the quadratic formula.
5) Any quadratic equation can be solved by completing the square or the quadratic formula.
6) The quadratic equation has two solutions, as shown in the quadratic formula $\boldsymbol{x}=\frac{-\mathbf{b} \pm \sqrt{\boldsymbol{b}^{2}-\mathbf{4 a c}}}{2 \boldsymbol{a}}$ the sign ( $\pm$ ) indicates that there are two roots.

$$
x_{1}=\frac{-b+\sqrt{b^{2}-4 a c}}{2 a}, \quad x_{2}=\frac{-b-\sqrt{b^{2}-4 a c}}{2 a}
$$

7) The expression $\boldsymbol{b}^{2}-\mathbf{4 a c}$ in the formula called the discriminate, which sign determines the type of roots such as:
i) $\boldsymbol{b}^{2}-\mathbf{4 a c}>\mathbf{0}$, the equation has two real roots.
ii) $\boldsymbol{b}^{2}-\mathbf{4 a c}=\mathbf{0}$, the equation has two equal real roots.
iii) $\boldsymbol{b}^{2}-\mathbf{4 a c}<\mathbf{0}$, the equation has no real roots.

## Examples:

1) Solve the equation $2 x^{2}-\boldsymbol{6} \boldsymbol{x}-\mathbf{1}=\mathbf{0}$ by completing the square.

## Solution:

$2 x^{2}-6 x-1=0 \quad \Longrightarrow \quad 2 x^{2}-6 x=1$
$\therefore x^{2}-3 x=\frac{1}{2}$, add $\left(-\frac{3}{2}\right)^{2}$ to both sides, then
$\therefore\left(x-\frac{3}{2}\right)^{2}=\frac{1}{2}+\frac{9}{4}$
$\left(x-\frac{3}{2}\right)^{2}=\frac{11}{4} \quad$, taking the square root to both sides we get $x-\frac{3}{2}$
$= \pm \frac{\sqrt{11}}{2}$
$\therefore x=\frac{3}{2} \pm \frac{\sqrt{11}}{2} \Longrightarrow x=\frac{3 \pm \sqrt{11}}{2}$
$\therefore$ The roots are: $x_{1}=\frac{3+\sqrt{11}}{2}, x_{2}=\frac{3-\sqrt{11}}{2}$
$\therefore$ The solution set is $S=\left\{\frac{3+\sqrt{11}}{2}, \frac{3-\sqrt{11}}{2}\right\}$.
2) Solve the equation $3 x^{2}-2 x-7=0$

Use the quadratic formula.

## Solution:

$$
3 x^{2}-2 x-7=0 \Longrightarrow a=3, b=-2, c=-7
$$

Applying the formula $x=\frac{-\mathrm{b} \pm \sqrt{b^{2}-4 a c}}{2 a}$
$x=\frac{-(-2) \pm \sqrt{(-2)^{2}-4(3)(-7)}}{2(3)}$
$=\frac{2 \pm \sqrt{4+84}}{6} \quad \therefore x=\frac{2 \pm \sqrt{88}}{6}$
$\therefore x=\frac{2 \pm 2 \sqrt{22}}{6} \quad \Longrightarrow \quad x=\frac{1 \pm \sqrt{22}}{3}$
$\therefore$ The roots are $x_{1}=\frac{1+\sqrt{22}}{3}, \quad x_{2}=\frac{1-\sqrt{22}}{3}$
$\therefore$ The solution set is $S=\left\{\frac{1+\sqrt{22}}{3}, \frac{1-\sqrt{22}}{3}\right\}$
3) Determine the type of roots for each of following equations by using the discriminant.
i) $3 x^{2}-4 x+1=0$
ii) $9 x^{2}-6 x+1=0$
iii) $5 x^{2}-2 x+1=0$

## Solution:

Using the discriminant $b^{2}-4 a c$ we get:
i) $3 x^{2}-4 x+1=0 \Longrightarrow a=3, b=-4, c=1$ then
$b^{2}-4 a c=(-4)^{2}-4(3)(1)$

$$
=16-12=4>0 \quad \Longrightarrow \text { roots are real. }
$$

ii) $9 x^{2}-6 x+1=0 \Longrightarrow a=9, b=-6, c=1$ then
$b^{2}-4 a c=(-6)^{2}-4(9)(1)$

$$
36-36=0 \Longrightarrow \text { roots are real and equal. }
$$

iii) $5 x^{2}-2 x+1=0 \Longrightarrow a=5, b=-2, c=1$ then $b^{2}-4 a c=(-2)^{2}-4(5)(1)$

$$
=4-20=-16<0 \Longrightarrow \text { roots are not real. }
$$

### 3.3.3 Simultaneous equations:

The equation in the form $\boldsymbol{a x}+\boldsymbol{b y}+\boldsymbol{c}=\mathbf{0}$ is called a linear equation in two variables (unknowns), it is called so because its graph is a straight line in the plane.

This equation has infinitely many solutions any ordered pair ( $x, y$ ) representing a point on this line is a solution to this equation.
A set of equations in the form:
$a_{1} x+b_{1} y+c_{1}=0$
$a_{2} x+b_{2} y+c_{2}=0$
is called a system of linear equations the solution set of such system is the ordered pair of numbers $(x, y)$ which satisfies both equations, there is usually just one solution to this system these equations are called simultaneous equations, because they both must be satisfied at the same time by the values of the variables $x$ and $y$ that we are looking for, that is the equations have to be solved together.

The main two algebraic methods of solving a system of linear (simultaneous) equations are the substitution method and elimination method as follows:

## I . Substitution method:

This done by finding an equivalent system to the original system of equations in which one of the equations contains only one variable then we substitute in the other equation and so on as illustrated in the following examples:

1) Solve the following system of linear equations by the substitution method:
$\left\{\begin{array}{c}3 x+4 y=8 \\ -2 x+y=13\end{array}\right.$
Solution:
Solving of the equation (ii) for y we obtain the equivalent system
$\left\{\begin{array}{l}3 x+4 y=8 \\ y=13+2 x\end{array}\right.$
Substitute for $y$ in equation (i) we obtain

The system $\begin{cases}3 x+4(13+2 x) & =8 \\ y=13+2 x & \ldots\end{cases}$
$\therefore$ Solving equation (i) for $x$ we obtain
$3 x+52+8 x=8 \Longrightarrow 11 x=8-52$
$\therefore 11 x=-44 \Longrightarrow x=-4$.
Then substitute in equation (ii) for
$\boldsymbol{x}=\mathbf{- 4}$ we obtain :

$$
\begin{aligned}
y=13+2 x \Longrightarrow y & =13+2(-4) \\
& =13-8 \\
& =5
\end{aligned}
$$

$\therefore$ the solution of this system is the ordered pair ( $-4,5$ ).
2) Solve the following system of linear equations by the substitution method:
$\left\{\begin{array}{c}2 x+6 y=5 \\ 4 x-3 y=0\end{array}\right.$

## Solution

$\left\{\begin{array}{c}2 x+6 y=5 \\ 4 x=3 y\end{array} \Longrightarrow\left\{\begin{array}{c}2 x+6 y=5 \\ x=\frac{3}{4} y\end{array}\right.\right.$
$\therefore\left\{\begin{array}{l}2\left(\frac{3}{4} y\right)+6 y=5 \\ x=\frac{3}{4} y \ldots \ldots \ldots \ldots\end{array}\right.$

If we simplify $2\left(\frac{3}{4} y\right)+6 y=5$ we obtain

$$
\begin{aligned}
\frac{3}{2} y+6 y=5 & \Longrightarrow 3 y+12 y=10 \Longrightarrow 15 y=10 \\
\therefore y=\frac{10}{15} & \Longrightarrow y=\frac{2}{3} . \text { substitute in (ii) }
\end{aligned}
$$

To obtain $x=\frac{3}{4}\left(\frac{2}{3}\right) \Longrightarrow x=\frac{1}{2}$
$\therefore$ the solution of the system is the ordered pair $\left(\frac{1}{2}, \frac{2}{3}\right)$.

## $\Pi$ - Elimination method:

This algebraic method uses the properties of equation to element one of these variables (unknowns) to produce a new equation with one unknown and then substitute in one of the equations of the system as illustrated in the following examples it is also called the addition or subtraction method, being it uses the operations of addition and subtraction of equations to eliminate the variable:

## Examples:

1) Solve the following system by the elimination method

$$
\left\{\begin{array}{r}
2 x+y=4  \tag{i}\\
x-y=5
\end{array}\right.
$$

## Solution:

Adding equation (i) and (ii) we obtain

$$
\begin{gather*}
2 x+y=4 \quad \ldots \ldots \ldots \ldots  \tag{i}\\
x-y=5 \quad \ldots \ldots \ldots \ldots  \tag{ii}\\
3 x \quad=9 \quad \Longrightarrow \quad x=3
\end{gather*}
$$

Substitute in either of the original equations for $x=3$ to obtain
$3-y=5 \longrightarrow-y=5-3 \longrightarrow-y=2$
$\therefore y=-2 \therefore$ the solution is the ordered pair (-2,3).
2) Solve the following system of linear equations:
3)
$\left\{\begin{array}{c}7 x+8 y=5 \\ 2 x+3 y=-5\end{array}\right.$

## Solution:

Multiply equation (i) by 2 and equation (ii) by (-7) and add

$$
\left\{\begin{array}{c}
14 x+16 y=10  \tag{i}\\
-14 x-21 y=35
\end{array}\right.
$$

$-5 y=45 \longrightarrow y=-9$
Substitute in equation (i) for $\boldsymbol{y}=-\mathbf{9}$

$$
\begin{aligned}
& 7 x+8(-9)=5 \longrightarrow 7 x-72=5 \\
& \therefore 7 x=5+72 \longrightarrow 7 x=77 \longrightarrow x=11
\end{aligned}
$$

$\therefore$ the solution is $(11,-9)$.

## 3.4 - The algebraic method of solving inequalities.

The in equality is defined as an open sentence consisting of two in equal sides each in the form of an algebraic expression connected by an inequality sign such as $>$ or $<$, such as $2 x-3>5$.
The in equality is called by the number of variables involved and its degree, such as:

$$
\begin{aligned}
& \boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}>\mathbf{0} \quad \boldsymbol{a} \neq \mathbf{0} \text { is a linear inequality } \\
& <0 \quad \text { in one variable } \\
& \boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} x+\boldsymbol{c}>\mathbf{0} \quad \boldsymbol{a} \neq \mathbf{0} \text { is a second degree } \\
& <\mathbf{0} \text { in equality in one variable. } \\
& \boldsymbol{a} \boldsymbol{x}+\boldsymbol{b} \boldsymbol{y}+\boldsymbol{c}>\mathbf{0} \quad \boldsymbol{a}, \boldsymbol{b} \neq \mathbf{0} \text { is a first degree in equality } \\
& <0 \text { in two variables. }
\end{aligned}
$$

It is called an inequality or an in equation Since it contains the verb in equal $(>,<)$ which means that the two sides are not equal.
A solution of an inequality is a value of the variable which makes the inequality a true statement, i.e. Satisfy it, the solution set is the set of all solutions.
Solving an inequality is finding its solutions (solution set). The algebraic method of solving inequalities is mainly similar to that of solving equations uses algebraic operations based up on the properties of real numbers and inequalities as follows:

### 3.4.1 - Linear inequality.

Is an inequality in the form $\boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}>\mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$, which is also called a first degree inequality in one variable.

We can solve linear inequalities just like linear equations by using the isolating method following a series of algebraic procedures using the following inequality properties:
i) $a>b \Longrightarrow a \pm c>b \pm c$, this states that you can add to or subtract anything to both sides of an inequality.
ii) $\quad a>b \Longrightarrow \begin{cases}a c>b c & \text { if } c>0 \\ a c<b c & \text { if } c<0\end{cases}$
iii) $\quad a<b \Longrightarrow \begin{cases}a c<b c & \text { if } c>0 \\ a c>b c & \text { if } c<0\end{cases}$
iv) This state that: you can multiply or divide both sides of an inequality by any positive number, but you have to reverse the inequality sign if the number is negative.

## Examples:

1) Solve the inequality $2(x+3) \geq(x+18)-2 x$

## Solution:

$2 x+6 \geq x+18-2 x$, (distributive law).
$2 x+6 \geq 18-x$
$2 x+x \geq 18-6 \quad \Longrightarrow 3 x \geq 12 \quad \Longrightarrow x \geq 4$
$\therefore$ Solution set algebraically is:
$S=\{x: x \geq 4\}$ and geometrically as shown in the figure (3.6)


Fig.(3.6)

And numerically is [ $4, \infty$ ).

### 3.4.2-Quadratic inequality.

Is an inequality in the form: $\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}>\mathbf{0}$, where $a, b$ and $c$ are real numbers and $\boldsymbol{a} \neq \mathbf{0}$, and $\mathbf{x}$ is the variable. it is also called a second-degree in equation in one variable (unknown). The main algebraic solving method of this type of inequality is the factoring method which based on the following inequality properties:
i) $\quad a b>0 \Longleftrightarrow\left\{\begin{aligned}+.+ & (a>0 \text { and } b>0) . \\ \text { or } \quad .- & (a<0 \text { and } b<0) .\end{aligned}\right.$

Which states that: the product of two positive numbers or two negative numbers is positive.
ii) $\quad a b<0 \Longleftrightarrow \begin{cases}+.- & (a>0 \text { and } b<0) . \\ \text { or }-.+ & (a<0 \text { and } b>0) .\end{cases}$

Which states that the product of a positive number and a negative number is negative number.

## Examples:

Solve the following inequalities algebraically

$$
\begin{aligned}
& \text { 1) } x^{2}-x-6>0 \Longrightarrow(x+2)(x-3)>0 \text {, ( factoring) } \\
& \begin{array}{llll} 
& + & + \\
\text { or } & - & & -
\end{array} \\
& \therefore(x+2)(x-3)>0 \Longleftrightarrow(x+2)>0 \quad \text { and }(x-3)>0 \\
& \text { or }(x+2)<0 \text { and }(x-3)<0
\end{aligned}
$$

i) $x+2>0 \Longleftrightarrow x>-2$ and $x-3>0 \Longleftrightarrow x>3$
ii) or $x+2<0 \Longleftrightarrow \mathrm{x}<-2$ and $x-3<0 \Longleftrightarrow x<3$

$$
\begin{aligned}
\mathrm{S} & =[\{x: x>-2\} \cap\{\mathrm{x}: \mathrm{x}>3\}] \cup[\{\mathrm{x}: \mathrm{x}<-2\} \cap[\{\mathrm{x}: \mathrm{x}<3\}] \\
& =\{x: x<-2\} \cup\{x: x>3\} ; \text { algebraically. }
\end{aligned}
$$

This is represented geometrically as shown in the figure (3.7)

gives (3, $\infty$ )

gives ( $-\infty,-2$ )


$$
\therefore \quad S=(-\infty,-2) \cup(3, \infty), \text { numerically. }
$$

Fig.(3.7)
2) $x^{2}-7 x+12 \leq 0$

## Solution:

$$
\begin{aligned}
& x^{2}-7 x+12 \leq 0 \Longrightarrow(x-3)(x-4) \leq 0,(\text { factoring }) \\
& \begin{aligned}
& +.- \\
\text { or } & -.+
\end{aligned} \\
& \therefore(x-3)(x-4) \leq 0 \Longleftrightarrow(x-3) \geq 0 \quad \text { and }(x-4) \leq 0 \\
& \text { or }(x-3) \leq 0 \text { and }(x-4) \geq 0 \\
& \text { i) } x-3 \geq 0 \Longrightarrow x \geq 3 \text { and } x-4 \leq 0 \Longrightarrow x \leq 4 \text {. } \\
& \text { ii) or } x-3 \leq 0 \quad \Longrightarrow x \leq 3 \text { and } x-4 \geq 0 \Longrightarrow x \geq 4 \\
& \therefore \mathrm{~S}=[\{x: x \geq 3\} \cap\{x: x \leq 4\}] \cup[\{x: x \leq 3\} \cap\{x: x \geq 4\}] \\
& =\{x: 3 \leq x \leq 4\} \text {, algebraically }
\end{aligned}
$$

This is represented graphically geometrically as shown in the figure (3.8)


Gives [3,4]


Gives $\emptyset$
$\therefore \mathrm{S}=[3,4]$ numerically.
Fig.(3.8)

### 3.4.3-Simultaneous inequalities:

Simultaneous inequalities are produced by combining two inequalities using the verbs (and, or) which are then replaced by ( $\cap, \cup$ ) respectively, the two inequalities in this system solved simultaneously but independently in the usual Manner as shown in the following examples:

## Examples:

Solve each of the following:

1) $2 x+1<5$ and $3 x+2 \leq-4$.

Solution: the solution set required is
$\mathrm{S}=\{x: 2 x+1<5$ and $3 x+2 \leq-4\}$ which is written as:
$\mathrm{S}=\{x: 2 x+1<5\} \cap\{x: 3 x+2 \leq-4\}$
This is solved simultaneously by putting them in the following form and solving them separately.

```
    \(2 x+1<5 \cap 3 x+2 \leq-4\)
    \(2 x<5-1 \cap 3 x \leq-4-2\)
    \(2 x<4 \cap 3 x \leq-6\)
\(\therefore x<2 \quad \cap \quad x \leq-2\)
\(\therefore \mathrm{S}=\{x: x<2\} \cap\{x: x \leq-2\}\), algebraically graphically.see the figure
```

(3.9)
$S=(-\infty,-2]$ numerically.


Fig.(3.9)
2) $-4 x-1 \geq x-6$ or $3 x+2<5 x+4$

Solution: the solution set required is written as:
$\mathrm{S}=\{-4 x-1 \geq x-6$ or $3 x+2<5 x+4\}$
Which is written as $S=\{-4 x-1 \geq x-6\} \cup\{3 x+2<5 x+4\}$
This is solved simultaneously putting the in the following form, and performing the operation on each separately as follows:
$-4 x-1 \geq x-6 \cup 3 x+2<5 x+4$
$-4 x-x \geq-6+1 \cup 3 x-5 x<4-2$
$-5 x \geq-5 \cup-2 x<2$
$\therefore x \leq 1 \cup x>-1$
$\therefore \mathrm{S}=\{x: x \leq 1\} \cup\{x: x>-1\}$ algebraically graphically. as shown in the figure (3.10)
$\therefore \mathrm{S}=\mathrm{R}$ all real numbers

$$
=(-\infty ; \infty) .
$$



Fig.(3.10)

## Notes:

1) The elements of the solution set of both equation and inequality differ according to the universal set of the variable.
2) If the universal set of the variable is $R$ (the set of real numbers) then the equation has solutions as its degree, while the inequality has infinity many solutions.

## Unit-4. Trigonometric concepts.

4.1 - Trigonometric ratios of an acute angle.
4.2 - Trigonometric ratios of complementary angle.
4.3 - Trigonometric ratios of a general angle.

## Unit: 4. Trigonometric concepts

## Introduction

Trigonometry is a branch of mathematics concerned with the study of the relationship between the parts of a triangle (sides, angles).
There are two types of trigonometry one is called plane trigonometry which is restricted to triangles lying in plane, the other is called spherical trigonometry which deals with triangles lying on sphere. The trigonometry has many applications in surveying, navigation, engineering, sound, light and electricity.

In this section we will introduce some concepts and relationships from plane trigonometry.

### 4.1 Trigonometric ratios of an acute angle:

In dealing with any right triangle, it will be convenient to denote the vertices as $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, denote the angles of the triangle as $\boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ and the sides opposite the angles as $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ respectively, as shown in the figure (4.1-a). With respect to angle $\boldsymbol{\theta}, \boldsymbol{a}$ will be called the opposite side and $\boldsymbol{b}$ will be called the adjacent side, with respect to angle $\boldsymbol{\beta}, \boldsymbol{a}$ will be called the adjacent side and $\boldsymbol{b}$ the opposite side $c$ will always be called the hypotenuse. as shown in the figure (4.1-b).


Fig.(4.1-a)


Fig.(4.1-b)

If now the right triangle is placed in a coordinate system so that angle $\boldsymbol{\theta}$ is in standard position, the point $\boldsymbol{B}$ the terminal side of angle $\boldsymbol{\theta}$ has coordinate ( $\boldsymbol{b}, \boldsymbol{a})$ and distance: $\boldsymbol{c}=\sqrt{\boldsymbol{a}^{2}+\boldsymbol{b}^{\mathbf{2}}}$. Then the trigonometric function of angle $\boldsymbol{\theta}$ may be defined in terms of the sides of the right triangle, as follows:
$\operatorname{Sin} \theta=\frac{a}{c}=\frac{\text { Opposite side }}{\text { hypotenuse }} \quad \cot \theta=\frac{b}{a}=\frac{\text { Adjacent side }}{\text { Opposite side }}$
$\operatorname{Cos} \theta=\frac{b}{c}=\frac{\text { Adjacent side }}{\text { hypotenuse }} \quad \sec \theta=\frac{c}{b}=\frac{\text { hypotenuse }}{\text { Adjacent side }}$

$$
\tan \theta=\frac{a}{b}=\frac{\text { Opposite side }}{\text { Adjaceent side }} \quad \csc \theta=\frac{c}{a}=\frac{\text { hypotenues }}{\text { Opposite sside }}
$$

### 4.2 Trigonometric ratios of complementary angles:

The acute angles $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ of the right triangle $\boldsymbol{A B C}$ are complementary, that is $\theta+\beta=90^{\circ}$. From Fig 3-A, we have

$$
\begin{array}{ll}
\operatorname{Sin} \beta=\frac{b}{c}=\cos \theta & \cot \beta=\frac{a}{b}=\tan \theta \\
\operatorname{Cos} \beta=\frac{a}{c}=\sin \theta & \sec \beta=\frac{c}{a}=\csc \theta \\
\operatorname{Tan} \beta=\frac{b}{a}=\cot \theta & \csc \beta=\frac{c}{b}=\sec \theta
\end{array}
$$

These relations associate the functions in pairs - sine and cosine, tangent and cotangent and cosecant - each function of a pair being called the corresponding co-Function of the complementary angle.

### 4.3 Trigonometric ratios of a general angle:

The axes divide the plane into four parts, called quadrants, which are numbered I, II, III, IV The numbered quadrants, together with the signs of the coordinates of a point in each are shown in Fig (8) P79.
The undirected distance r of any point $P(x, y)$ from the origin, called the distance of P or the radius vector of $P$ is given by

$$
R=\sqrt{x^{2}+y^{2}}
$$

Thus, with each point in the plane, we associate three numbers $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{r}$. i)Angles in standard position:

With respect to a rectangular coordinate system, an angle is said to be in standard position when its vertex is at the origin and its initial side coincides with the positive $\boldsymbol{x}$ - axis.

An angle is said to be a first quadrant angle or to be in the first quadrant if, when in standard position, its terminal side falls in that quadrant.

Similar definitions hold for the other quadrants. For example, the angles $30^{\circ}, 59^{\circ}$ and $-330^{\circ}$ are first quadrant angles; $119^{\circ}$ is a second quadrant angle; $-119^{\circ}$ is a third quadrant angle; $-10^{\circ}$ and $710^{\circ}$ are fourth quadrant angles. as shown in the figure (4.2)



Fig.(4.2)

Two angles which, when placed in standard position, have coincident terminal sides are called conterminal angles. For example, $30^{\circ}$ and $-330^{\circ}$; $-10^{\circ}$ and $710^{\circ}$ are pairs of coterminal angles There are an unlimited number of angles coterminal with a given angle. (see problem 4)
The angles $\mathbf{0}^{\mathbf{0}}, \mathbf{9 0}^{\mathbf{o}}, \mathbf{1 8 0}^{\mathbf{o}}, \mathbf{2 7 0}^{\mathbf{o}}$, and all angles conterminal with them are called quadrant angles.
ii) Trigonometric ratios of a general angle.

Let $\boldsymbol{\theta}$ be an angle (not quadrant) in standard position and let $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ be any point, distance from the origin on the terminal side of the abscissa, ordinate and distance of $\boldsymbol{P}$, as follows:

$$
\operatorname{sine} \theta=\sin \theta=\frac{\text { Ordinate }}{\text { Distance }}=\frac{y}{r}
$$

$$
\begin{aligned}
& \text { Cosine } \theta=\cos \theta=\frac{\text { Abscissa }}{\text { Distance }}=\frac{x}{r} \\
& \text { tangent } \theta=\tan \theta=\frac{\text { ordinate }}{\text { Abscissa }}=\frac{y}{x}
\end{aligned}
$$

As an immediate consequence of these definitions, we have the so-called Reciprocal:see the figure (4.3)

$$
\begin{array}{ll}
\text { cosecant } \theta=\csc \theta=\frac{1}{\sin \theta} & \csc \theta=\frac{\text { Distance }}{\text { ordinate }}=\frac{x}{y} \\
\text { Secant } \theta=\sec \theta=\frac{1}{\cos \theta} & \sec \theta=\frac{\text { Distance }}{\text { Abscissa }}=\frac{r}{x} \\
\text { cotangent } \theta=\cot \theta=\frac{1}{\tan \theta} & \cot \theta=\frac{\text { Abscissa }}{\text { ordinate }}=\frac{x}{y}
\end{array}
$$

$$
\operatorname{Sin} \theta=\frac{1}{\csc \theta}, \quad \cos \theta=\frac{1}{\sec \theta}, \quad \tan \theta=\frac{1}{\cot \theta}
$$





$$
P(x, y)
$$


$P(x, y)$

Fig.(4.3)

It is evident from the figures that the values of the trigonometric functions of $\theta$ change as $\theta$ changes. In problem 5 it is shown that the values of $P$ on its terminal side.
iii) Algebraic signs of the ratios:

Since $r$ is always positive, the signs of the ratios in the various quadrants depend upon the signs of $x$ and $y$. To determine these signs, one may visualize the angle in standard position or use some device as shown in the figure (4.4), in which only the ratios having positive signs are listed (see problem 6).

When an angle is give $\mathbf{n}$, its trigonometric ratios are uniquely determined. When, however, the value of one ratios of an angle is given, the angle is not uniquely determined. For example, if $\sin \theta=\frac{1}{2}$ then $\theta=30^{\circ}$, $150^{\circ}, 390^{\circ}, 510^{\circ}, \ldots$. In general, two possible positions of the terminal side are found - for example, the terminal sides of 30 and 150 in the above illustration. The exceptions to this rule occur when the angle is quadrantal.


Fig.(4.4)
iv) Trigonometric ratios of quadrantal angles. For a quadrantal angle, the terminal side coincides with one of the axes. A point $P$, distinct from the origin, on the terminal side has rather $x=0, y \neq 0$ or $x \neq 0, y=0$. In either case, two of the six ratios will not be defined. For example, the terminal side of the angle 0 coincides with the positive $x$ - axis and the ordinate of $P$ is 0 . Since the ordinate occurs in the denominator of the ratio defining the cotangent and cosecant, these ratios are not defined. Certain authors indicate this by writing $\boldsymbol{\operatorname { c o t }} \mathbf{0}=\infty$ and others write $\boldsymbol{\operatorname { c o t }} \mathbf{0}= \pm \infty$ , the following results are obtained.table (4.1) illustrates the trigonometric
ratios of angles $0^{\circ}, 90^{\circ}, 180^{\circ}$ and $270^{\circ}$. and table(4.2) illustrates trigonometric ratios of $30^{\circ}, 45^{\circ}, 60^{\circ}$. see the figure (4.5).

## Table.(4.1)

| Angle $\theta$ | $\operatorname{Sin} \theta$ | $\operatorname{Cos} \theta$ | $\operatorname{Tan} \theta$ | $\operatorname{Cot} \theta$ | $\operatorname{Sec} \theta$ | $\csc \theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $0^{\circ}$ | 0 | 1 | 0 | $\pm \infty$ | 1 | $\pm \infty$ |
| $90^{\circ}$ | 1 | 0 | $\pm \infty$ | 0 | $\pm \infty$ | 1 |
| $180^{\circ}$ | 0 | -1 | 0 | $\pm \infty$ | -1 | $\pm \infty$ |
| $270^{\circ}$ | -1 | 0 | $\pm \infty$ | 0 | $\pm \infty$ | -1 |

Table(4.2)

| Angle $\theta$ | $\operatorname{Sin} \theta$ | $\operatorname{Cos} \theta$ | $\operatorname{Tan} \theta$ | $\operatorname{Cot} \theta$ | $\operatorname{Sec} \theta$ | $\csc \theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $30^{\circ}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ | $\frac{2 \sqrt{3}}{3}$ | 2 |
| $45^{\circ}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | 1 | $\sqrt{2}$ | $\sqrt{2}$ |
| $60^{\circ}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{\sqrt{3}}{3}$ | 2 | $\frac{2 \sqrt{3}}{3}$ |



Fig.(4.5)
vi) Tangent Cosine and sine ratios for obtuse angles.

Here is a very simple guide for finding the tangent cosine and sine ratios for obtuse angles (angles over $90^{\circ}$ but less than $180^{\circ}$ ), using the following trigonometric relationship.
$\operatorname{Tan} \alpha=-\tan \left(180^{\circ}-\alpha\right)$, so $\tan 145^{\circ}=-\tan 35^{\circ}$
$\cos \alpha=-\cos \left(180^{\circ}-\alpha\right)$, so $\cos 130^{\circ}=-\cos 50^{\circ}$
$\operatorname{Sin} \alpha=+\sin \left(180^{\circ}-\alpha\right)$, so $\sin 120^{\circ}=+\sin 60^{\circ}$

## Unit: 5. Analytic geometry concepts

Introduction
5.1- Rectangular (Cartesian) co-ordinate system:
5.1.1-The point in one -dimension
5.1.2-The point in two- dimensions
5.1.3-The point in three - dimensions:
5.2-Other co-ordinate systems:
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## Unit 5: Analytic geometry concepts

## Introduction

Analytic geometry which is also called co-ordinate geometry is a branch of mathematics which studies geometry through the use of algebra. The subset was first introduced by the French philosopher and mathematician (Rene Descartes, 1596-1650) who used the idea of Cartesian co-ordinates to determine the position of any point in the plane. This type of geometry has been begun by those who have had practice in drawing algebraic graphs through the use of the fundamental principle of analytic geometry which is based up on the concept of one to one correspondence between the points on the real line and the real numbers which makes it possible to represent any set of real numbers by a picture or graph on the real line, conversely we can describe or analyze algebraically any set of points on the real line by considering the set of numbers corresponding to these points, this idea of correspondence is called the fundamental principle of analytic geometry according to this we think of the real numbers as points on the real line and vice verso which opened the way in front of the co-ordinate techniques enabling arithmetical and algebraically work to be brought in to geometry which makes a link between algebra and geometry enabling proofs by pure and abstract geometry to be replaced by algebraic proofs, and the properties of curves and geometric figures to be investigated algebraically, which will enable us to describe curves in the plane by means of algebraic equations.

## 5.1 - Rectangular (Cartesian) co-ordinate system:

Determining and locating the position of a point geometrically and representing it algebraically were done using the idea of the fundamental principle of analytic geometry which was done by establishing a co-ordinate system on the real line. The form of representation differs according to the space in which the point moves, as follows:

### 5.1.1 The point in one-dimension:

If the point moves on a straight line, then the motion is called a linear motion or a one - dimensional type. the position of the point in this case can be determined by using the idea of one-to-one correspondence between the real numbers and the points on the line. this is done as follows draw a
straight line and choose a reference point (0) on it and mark the real numbers on this line by choosing a unit length the numbers on both sides of the reference point (0) are labeled positive (+ve) or negative (-ve) according to the direction following a mathematical convention as shown by the following figure (5.1) the position of a point is represented by a dot (.) on the line and described by the number associated with it on this line, as shown.


Fig (5.1)
This line is called a one - dimensional co- ordinate system.
The points in this system are written as $A\left(x_{1}\right), B\left(x_{2}\right), C\left(x_{3}\right), \ldots$
Such as $\mathrm{A}(2), \mathrm{B}(4), \mathrm{C}(-2), \mathrm{D}(-4)$ Indicating the points on this line

## Notes:

from the figure (5.2) we noted that:

1) The real line is called the horizontal or $x$ - axis which extends to infinity from both sides.
2) The reference point ( 0 ) is put in the middle of the straight line representing the zero point and is called the origin beining the original point from which the motion of the point is measured.
3) According to a mathematical convention the numbers to the right of the origin ( 0 ) are positive $(+\mathrm{ve})$ and negative $(-\mathrm{ve})$ to its left while the origin $(0)$ itself is neither $(+$ ve $)$ nor $(-$ ve $)$ it is a neutral point.


Fig.(5.2)
4) The point in this co-ordinate system (one - dimension) is written algebraically as $(x)$, where $(P)$ means point and $(x)$ is called the coordinate of the point $(P)$ which indicates its position on the real line.
5) The directed distance from the point $A\left(x_{1}\right)$ to the point $B\left(x_{2}\right)$ in one dimension is denoted by $\overline{A B}$ or $|A B|$, see the figure (5.3)


Fig (5.3)
and is defined by:
$\overline{\boldsymbol{A B}}=\overline{\mathbf{0 B}}-\overline{\mathbf{0 A}}=\boldsymbol{x}_{\mathbf{2}}-\boldsymbol{x}_{\mathbf{1}}$, and it is may be positive (+ve) or negative (- ve) according to the position of the points on the line there are seven possible cases as shown in the figures (5.4-a, ...,g)


$$
\begin{aligned}
\overline{A B} & =\overline{0 B}-\overline{0 A} \\
& =5-2=3
\end{aligned}
$$

Fig.(5.4-a)


Fig.(5.4-c)


Fig.(5.4-e)


$$
\begin{aligned}
\overline{A B} & =\overline{0 B}-\overline{0 A} \\
& =2-5=-3
\end{aligned}
$$

Fig.(5.4-b)


Fig.(5.4-d)


Fig.(5.4-f)

$\overline{A B}=\overline{O B}-\overline{O A} \quad \boldsymbol{A}$ and $\boldsymbol{B}$ are the same points

$$
\left.\begin{array}{rl} 
& =5-5 \\
\text { or } & =-5-(-5)=0
\end{array}\right\}
$$

i.e., they are coincident
i.e., $\overline{O A}=\overline{0 B}$

Fig.(5.4-g)
6) The undirected distance between the points $\mathrm{A}\left(x_{1}\right), \mathrm{B}\left(x_{2}\right)$ which is sometimes called the distance from either of these points to the other is defined as:

$$
|\overline{A B}|=|\overline{B A}|=\left|x_{2}-x_{1}\right|=\left|x_{1}-x_{2}\right| .
$$

We observe that the absolute value sign is used in this definition indicating that this distance is never negative it is always positive because it gives the length of the line segment $A B$ or $B A$ which are always positive

### 5.1.2 The point in two- dimensions:

If the point is free to move in a plane, then the motion is called a plane motion or a two- dimensional type. The position of the point in two dimensions (in plane) is determined by using the idea of one-to-one correspondence between the set of ordered pairs of real numbers $(\boldsymbol{x}, \boldsymbol{y})$ and the set of points in the plane. The procedure of establishing this type of correspondence starts by constructing a co-ordinate system by drawing two intersecting lines at right angles to each other (that is they are perpendicular to each other) at their point of intersection, one of these two lines is horizontal and called the $\boldsymbol{X}$ - axis and the other is vertical and called $\boldsymbol{Y}$ axis both are called rectangular co-ordinate axes, their point of intersection is called the origin. on each axis establish a co- ordinate system that have a common origin $(\mathbf{O})$ as done in the case of one-dimension.

The process of determining and locating and identifying points in the plane takes two directions, one of them is concerned with the algebraic
representation of the point which takes the direction from a point to an ordered pair of real numbers $(\boldsymbol{x}, \boldsymbol{y})$.

Now let ( $\boldsymbol{P}$ ) be any point in the plane given geometrically as a dot (.) in the plane. draw two lines $\boldsymbol{L}_{\boldsymbol{1}}$ and $\boldsymbol{L}_{\boldsymbol{2}}$ through $\boldsymbol{P}$ one line perpendicular to each axis as figure (5a). Then the number corresponding to this point at the intersection of $\boldsymbol{L}_{1}$ and the $\mathbf{X}-\mathbf{a x i s}$ is called the x-coordinate of $\boldsymbol{P}$ and the number corresponding to the point at the intersection of the line $\boldsymbol{L}_{2}$ and the $y$-axis is called the y-coordinate of $\boldsymbol{P}$. Then the point $\boldsymbol{P}$ is associated with the ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ of real numbers and written algebraically in the form $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$. The other direction is the opposite to the first one which takes the direction from an ordered pair to a point, this direction is concerned with process of identifying the position of the point in the plane in terms of its coordinates which is called the process of plotting of the graph of the point $\boldsymbol{P}$.

Now let the point be given as an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ of numbers, to determine the position of $\boldsymbol{P}$ that is to plot the point $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$, determine first the positions of the numbers $\boldsymbol{x}$ and $\boldsymbol{y}$ on the $\mathbf{x}$ and $\mathbf{y}$ axes respectively and then draw lines $\boldsymbol{L}_{\boldsymbol{1}}$ and $\boldsymbol{L}_{\boldsymbol{2}}$ perpendicular to the axes at these numbers $\boldsymbol{x}$ and $\boldsymbol{y}$ then as shown in the figure (5.5-a), the point of intersection of $\boldsymbol{L}_{\boldsymbol{1}}$ and $\boldsymbol{L}_{2}$ is the required point $\boldsymbol{P}$ whose coordinates are $\boldsymbol{x}$ and $\boldsymbol{y}$. Then the ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ is associated with the point $\boldsymbol{P}$ in the plane. We have thus described a procedure that establishes a one - to one correspondence between the ordered pairs of real numbers and the points in the plane, which indicates that to each point in the plane corresponds one and only one ordered pair of real numbers and to each ordered pair corresponds one and only one point in the plane.

Under this association, each point in plane is identified with an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ of real numbers, conversely each ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ is identified with a single point whose coordinate are $\boldsymbol{x}$ and $\boldsymbol{y}$.

We will suggest here that the two directions of the previous process of determining and locating and identifying the position of any point in the plane can be done by an equivalent different way in terms of its directed perpendicular distances from the two co-ordinate axes as follows:

In the first case which is concerned with the algebraic representation of the point which is usually gives as a dot (.). In the plane we determine its perpendicular distances from the two axes by the use of the squares in the graph paper as shown in the figure (5.5b), call these distances $\boldsymbol{x}$ and $\boldsymbol{y}$ then the point is written as $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$, this shows that; if to reach the point $\boldsymbol{P}$ in the plane from the origin $\mathbf{O}$ you have to move a directed distance of $\boldsymbol{x}$ units along the $\boldsymbol{X}$ - axis then a directed distance of $\boldsymbol{y}$ along the $\boldsymbol{Y}$ - axis, conversely. If we are given the pair $(\boldsymbol{x}, \boldsymbol{y})$ we move from $\mathbf{O} \boldsymbol{x}$ units on the $\boldsymbol{X}$ - axis then we move $y$ units on the $\boldsymbol{Y}$ - axis the end point then will be the point $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$.


Fig.(5.5a)


Fig.(5.5b)

## Notes:

1) The horizontal line is called the $\mathbf{X}$ - axis, while the vertical line is called the $\mathbf{Y}$ - axis, and the two together are called the rectangular coordinate axes i.e., axes which are mutually perpendicular.
2) According to a mathematical convention the $\mathbf{X}$ - axis is considered with the positive (+ ve ) direction to the right of ( $\mathbf{O}$ ) and negative ( -ve ) to the left and the $\mathbf{Y}$ - axis with the positive (+ve) direction upward and negative (-ve) downward.
3) The co-ordinate axes divide the plane into four regions (parts) called quadrants which are numbered $\mathbf{I}, \Pi, \Pi \Pi, I V$ as shown in the figure (5.6).


Fig.(5.6)
This plane is then called the Cartesian co-ordinate plane, or simply $\boldsymbol{x} \boldsymbol{y}$ plane.
4) The axes and the plane extend to infinity from all sides.
5) The point in this co-ordinate system (two-dimensions) is written algebraically as $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ (and not as $=(\boldsymbol{x}, \boldsymbol{y})$ ) where $\boldsymbol{p}$ means point and $(\boldsymbol{x}, \boldsymbol{y})$ is an ordered pair of numbers indicating the co-ordinates the geometric figure of $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ is called the graph of the ordered pair $(\boldsymbol{x}, \boldsymbol{y})$.
6) Due to a mathematical convention, the $x$-coordinate appears first in the pair and $y$-coordinate is the second.
Thus, it is called an ordered pair due to the order of co-ordinates indicated above.
7) In the notation $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ the $x$ and $y$ represent the directed distances of the point $\boldsymbol{p}$ from the axes, as shown in the figure (5.7) the distance of $\boldsymbol{p}$ from the $\mathbf{Y}$-axis is called the $\boldsymbol{x}$-coordinate or abscissa, the distance of $\boldsymbol{p}$ from the $\mathbf{X}$ - axis is called the $\mathbf{y}$-coordinate or ordinate, the two distances taken together are called the Cartesian co-ordinates of ( $\boldsymbol{p}$ ), after (Descartes, 1596-1650).


Fig.(5.7)
8) The signs of the co-ordinates of a point differs according to the quadrant in which the point is lying as shown in the figure (5.8).


Fig.(5.8)
9) Any point on the $\mathbf{X}$-axis its $y$-coordinate is zero and written in the form of $\boldsymbol{P}(\boldsymbol{x}, \mathbf{0})$.
Any point on the $\mathbf{Y}$-axis its $x$-coordinate is zero and written in the form $\boldsymbol{P}(\mathbf{0}, \boldsymbol{y})$. as shown in the figure (5.9).
The origin $\mathbf{O}$ has coordinates $(\mathbf{0}, \mathbf{0})$ and written in the form $\mathbf{O}(\mathbf{0}, \mathbf{0})$


Fig(9)
10) Two points in the plane are the same called coincide points if and only if (iff) they have the same $\boldsymbol{x}$-coordinates and the same $\boldsymbol{y}$-coordinates thus $\boldsymbol{P}_{\mathbf{1}}\left(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{y}_{\mathbf{1}}\right)$ and $\boldsymbol{P}_{\mathbf{2}}\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}\right)$ are the same point iff $\boldsymbol{x}_{\boldsymbol{1}}=\boldsymbol{x}_{\mathbf{2}}, \boldsymbol{y}_{\mathbf{1}}=\boldsymbol{y}_{\mathbf{2}}$. 11) Locating a point in the coordinate plane corresponding to an ordered pair of real numbers is called " plotting the point ", the geometric figure representing the location (position) of the point $\boldsymbol{P}\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)$ in the plane is called the graph of the ordered pair $\left(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{y}_{\mathbf{1}}\right)$.
12) When a set of points, whose co-ordinates are given, are to be plotted, a suitable scale is chooses and marked on both axes, a square paper or graph paper is used in this case as shown in the following example.

## Example:

Plot each of the following points
$P_{1}(3,0), P_{2}(-4,0), P_{3}(0,2), P_{4}(0,-3), P_{5}(4,5), P_{6}(-3,4), P_{7}(-4,-3)$, $P_{8}(3,-4)$, on a graph paper.
Solution: the points are ploted as shown in the figure (5.10).


Fig(5.10)
13) The notation $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ denotes a general point which represents the coordinates of any point in the plane i.e., $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ is not a specific point which means its position is not fixed, while the notation $\boldsymbol{P}\left(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{y}_{\mathbf{1}}\right)$ or $\boldsymbol{P}\left(\boldsymbol{x}^{\prime}, \boldsymbol{y}^{\prime}\right)$ denotes a special point, using this suffix notation makes it possible to represent a specific point which have a known fixed position such as:
$P_{1}(2,3), P_{2}(-3,4), P_{3}(-2,-5), \ldots$

### 5.1.3 The point in three - dimensions:

If the point is free to move in space, then the motion is called a space motion or a three-dimensional type.

The position of the point in three-dimensions (in space) is determined by using the idea of one-to-one correspondence between the set of ordered triples of numbers $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and points in space the procedure of establishing this type of correspondence starts with constructing a coordinate system by drawing three lines each at right angles to the other two interesting at a common point $\mathbf{O}$. Two of these are horizontal and called $\mathbf{X}-$ axis and $\mathbf{Y}$ - axis the third is vertical and called $\mathbf{Z}$ - axis on each line establish a co-ordinate system that have a common origin O as done in the case of one and two dimensions, and the three together are called the rectangular coordinate axes.
i.e., axes which are mutually perpendicular, their point of intersection is called the origin $\mathbf{O}$.

The process of determining and locating and identifying points in space is as indicated in case of two - dimensions take two directions, one of them is concerned with algebraic representation of the point that is from a point to an ordered triple of real numbers $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$.

Now let $\boldsymbol{P}$ be any point in space given geometrically as a dot (.) in the space through $\boldsymbol{P}$ draw three planes one plane perpendicular to each coordinate axis as shown in the figure (5.11).

These three planes intersect the three axes at points corresponding to numbers $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$, these numbers are then called the $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}-$ coordinates of $P$. There for the point in space is associated with the
ordered triple of real numbers $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and written algebraically in the form $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$.


Fig.(5.11)

The other direction is from an ordered triple of numbers $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ to a point $\boldsymbol{P}$ in space, Thus it is concerned with the identification of the position of the point in space in terms of its co-ordinates ( $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ ) Now let the point $\boldsymbol{P}$ be given as an ordered triple $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of numbers, determine first the positions of the co-ordinates $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ on the axes and then draw three planes each is perpendicular to one axis at the corresponding co-ordinate, then $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ will be the point of intersection of these three planes.
This indicates that the ordered triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of real numbers is associated with the point $\boldsymbol{P}$ in space.
We have thus described a procedure of establishing a one - to one correspondence between the points in three - dimensional space and the set of ordered triples of real numbers which states that to each point in space corresponds one and only one ordered triple of real numbers, and to each ordered triple corresponds one and only one point in the space, under this
association, each point in space is identified with an ordered triple of real numbers $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ conversely, each ordered triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is identified with a single point whose co-ordinates are $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$.

## Notes:

1) The three lines in the three - dimensional system are called the coordinate axes, two of them are horizontal in the same plane the other is vertical and perpendicular to this plane. The three axes are mutually perpendicular to each other.
2) The point of intersection of these three axes is called the origin $\mathbf{O}$.
3) The axes are usually identified by letters and most frequently are written as $\mathbf{X}$ - axis, $\mathbf{Y}$ - axis for the two horizontal axes and $\mathbf{Z}$ - axis for the vertical one.
4) There are two types of the co-ordinate axes system as shown in the figure ( $5.12, \mathrm{a}, \mathrm{b}$ ) one of them is called left - handed co-ordinate system in which the $\mathbf{X}-$ axis is considered as the second, and the $\mathbf{Z}$ - axis as the third. If the $\mathbf{X}-\mathbf{a x i s}$ and $\mathbf{Y}-$ axis are interchanged the resulting co-ordinate system is then called a right - handed system. It is generally most convenient to choose mutually perpendicular axes, they are then called " rectangular " otherwise they are called "oblique".


Fig. $(5,12 a)$
Fig.(5.12b)
5) According to a mathematical convention the signs of directions on the axes are as shown in the figure (5.13).


Fig.(5.13)
6) In drawing a three - dimensional co-ordinate axes on a two dimensional piece of paper as shown in the figure (5.14) we customarily represent the $\boldsymbol{Y}$-axis by a line for which the angle ( $\boldsymbol{X O Y}$ ) is about $120^{\circ}$ and the unit of length on the $\boldsymbol{Y}$ - axis is taken to be about two - thirds $\left(\frac{2}{3}\right)$ of that on the $\boldsymbol{X}$ and $\boldsymbol{Z}$ axes this is to make the picture appears as if in space.


Fig.(5.14)
7) The three - coordinate axes taken in pairs determine three planes as shown in the figure (5.15) the $\boldsymbol{X} \boldsymbol{Y}$ - plane contains the $\mathbf{X}$ and $\mathbf{Y}$ axes, the $\boldsymbol{X Z}$ - plane contains the $\boldsymbol{X}$ and $\boldsymbol{Z}$ axes, the $\boldsymbol{Y Z}$ - plane contains the $\boldsymbol{Y}$ and $\boldsymbol{Z}$ axes.


Fig.(5.15)
8) Each of the three axes and three planes extends to infinity from all sides and perpendicular to each other.
9) Since each co-ordinate plane divides the space into two parts then the three -coordinate planes together divide the whole space into eight parts (regions) called octants, they are numbered as $\mathbf{I}, \Pi, I \Pi, I V, \mathbf{V}, \mathbf{V I}, \mathbf{V I I}$, and VIII. as shown in the figure (15) the first octant (number I) is the octant whose bounding edges are the positive (+ve) $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ axes then $\mathbf{I}, \Pi$, II and IV lie above the $\boldsymbol{X Y}$-plane in counter - clockwise order about $(\boldsymbol{O Z})$, octants number V, VI, VII, and VIII lie below the XY-plane, number V lying under number I.
10) The point in this co-ordinate system (three-dimensions) is written algebraically as $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ where $\boldsymbol{P}$ means point and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is an ordered triple of real numbers indicating the Cartesian co-ordinates of this point, the geometric figure of this point $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is called the graph of the ordered
triple $(\boldsymbol{x}, \boldsymbol{y}, \mathbf{z})$, and it is called an ordered triple due to the order of coordinates in the notation $(x, y, z)$ which it came according to a mathematical convention.
11) In the notation $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ the $\boldsymbol{x}$ and $\boldsymbol{y}$ and $\boldsymbol{z}$ represent the directed distances of the point $\boldsymbol{P}$ from the three mutually perpendicular co-ordinate planes, then $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ are called the Cartesian co-ordinates of the point $\boldsymbol{P}$. as shown in the figure (5.16) $\boldsymbol{x}$-coordinate of $\boldsymbol{P}$ denotes the perpendicular distance from $\boldsymbol{P}$ to $\boldsymbol{Y Z}$-plane the $\mathbf{y}$-coordinate of $(\boldsymbol{P})$ denotes the perpendicular distance from $P$ to $X Z$-plane and the $Z$ - coordinate of $P$ denotes the perpendicular distance from $(P)$ to $X Y$ - plane.


Y
Fig.(5.16)
12) The signs of the co-ordinates of a point are determined by the octant in which the point lies. Table (5.1) shows the signs for the eight octants in the left - handed co-ordinate system.

Table.(5.1).

| CoOrdinate | Octants |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | I | $\Pi$ | IП | IV | V | VI | VII | VIII |
| X | + | + | - | - | + | + | - | - |
| Y | + | - | - | + | + | - | - | + |
| Z | + | + | + | + | - | - | - | - |

13) Any point on the $\mathbf{X}$-axis its $\boldsymbol{y}$ and $\mathbf{z}$ co-ordinates are zero and written in the form $\boldsymbol{P}(\boldsymbol{x}, \mathbf{0}, \mathbf{0})$. Any point on the $\mathbf{Y}$-axis it's $\boldsymbol{x}$ and $\mathbf{z}$ co-ordinates are zeros and written in the form $\boldsymbol{P}(\mathbf{0}, \boldsymbol{y}, \mathbf{0})$. Any point on the $\mathbf{Z}$-axis its $\boldsymbol{x}$ and $\boldsymbol{y}$ co-ordinates are zero and written in the form $\boldsymbol{P}(\mathbf{0}, \mathbf{0}, \boldsymbol{z})$.
14) Any point on the $\boldsymbol{X} \boldsymbol{Y}$-plane its z-coordinate is zero and written as $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \mathbf{0})$. Any point on the $\mathbf{X Z}$-plane its $\mathbf{y}$-coordinate is zero and written as $\boldsymbol{P}(\boldsymbol{x}, \mathbf{0}, \boldsymbol{z})$ Any point on the $\boldsymbol{Y Z}$-plane its $\boldsymbol{x}$-coordinate is zero and written as $\boldsymbol{P}(\mathbf{0}, \boldsymbol{y}, \boldsymbol{z})$.
15) The process of determining the position of the point $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ in space is called " plotting " the point, the geometric figure dot (.), representing this location ( position )of this point is called the graph of the ordered -triple of numbers $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$.
16) To plot the point $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ on a graph paper draw the co-ordinate axes as indicated in the previous note (6) then to reach the point $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ from the origin $\boldsymbol{O}$ move a distance of $\boldsymbol{x}$-units along the $\boldsymbol{X}$-axis and then a distance of $\boldsymbol{y}$ units along (or parallel) to $\boldsymbol{Y}$-axis , and then a distance of $\mathbf{z}$ units along (or parallel ) to the Z-axis, notice that the distances $\boldsymbol{x}, \boldsymbol{y}$ and $\mathbf{z}$ are directed distances as shown in the figure (5.17).


Fig.(5.17)
17) The notation $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ denotes a general point which can be any point in space the suffix notation $\boldsymbol{P}\left(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{y}_{\mathbf{1}}, \boldsymbol{z}_{\mathbf{1}}\right)$ denotes a special or a specific point which has a specific position as:
$P_{1}(0,2,3), P_{2}(-1,2,-3) P_{3}(4,-2,1)$.
18) Two points $\boldsymbol{P}_{1}$ and $\boldsymbol{P}_{\mathbf{2}}$ are symmetric with respect to a plane if and only if (iff) that plane bisects the line segment $\boldsymbol{P}_{\mathbf{1}} \boldsymbol{P}_{\mathbf{2}}$ and is perpendicular to it. For example:

- The points $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and $(\boldsymbol{x}, \boldsymbol{y},-\boldsymbol{z})$ are symmetric with respect to the XY-plane.
- The points $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and $(\boldsymbol{x},-\boldsymbol{y}, \boldsymbol{z})$ are symmetric with respect to the XZ-plane.
- The points $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and $(-\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ are symmetric with respect to the YZ-plane.

19) The two - dimensional analytic geometry is called plane analytic geometry or plane co-ordinate geometry being concerned with the study of the geometric concepts and relationships in the plane.

- The three - dimensional analytic geometry is called space analytic geometry or solid analytic geometry being concerned with the study of geometric concepts and relationship in space.

20) If R is the set of real numbers, then:
i) $\quad \boldsymbol{R}$ is identified with one - dimensional space and called simply 1space.
ii) $\quad \boldsymbol{R} \times \boldsymbol{R}$ is identified with two - dimensional space and called simply 2 - space.
iii) $\boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$ is identified with three - dimensional space and called simply $3-$ space.

### 5.2. Other co-ordinate systems:

Rectangular (Cartesian) co-ordinate system furnishes only one method of locating a point. The need of calculus for simplifying the methods of problem-solving involving solids makes it necessarily to establish new coordinate systems in addition to the Cartesian rectangular co-ordinate system to facilitate the methods of integration by the use of an appropriate transformation formulas. These systems are: polar - cylindrical and spherical co-ordinate systems.

### 5.2.1 - polar co-ordinate system

Polar co-ordinates are introduced to facilitate the evaluation of the double integral through the use of transformations suggested by this type of point representation. There are two forms of polar co-ordinate representation according to the space in which the point lies:
a) Polar co-ordinates in (plane) two - dimensions in the previous rectangular (Cartesian) co-ordinate system the position of a point in the plane identified in terms of its directed distances from two mutually perpendicular axes namely $X$ and $Y$ axes. In the polar co-ordinate system the position of the point in plane is identified in terms of its directed distance from a fixed reference point called the pole, and its direction from a fixed line called the polar axis, passing through the pole, then as shown in the figure (5.18) if $P$ is a point in plane and 0 is the pole and $(r)$ is the distance from 0 to $P$ is $r$ and $O A$ is a fixed line and $\theta$ be the angle through wish a line rotates from the position $0 A$ into the position $O P$, then $(r, \theta)$ are called the polar co-ordinates of $P$ and $P$ is denoted by the notation $P(r, \theta)$.


Fig.(5.18)

## Notes:

1) The origin point $\boldsymbol{O}$ is called the pole.
2) The fixed line is called the polar axis.
3) The angle $\boldsymbol{\theta}$ is called the vectorial angle or the polar angle, $\boldsymbol{\theta}$ being (+ve) positive or (-ve) negative according as the direction of rotation is counter - clockwise or clockwise.
4) The distance between the pole $\mathbf{O}$ and the point $\boldsymbol{P}$ is denoted by ( $\boldsymbol{r}$ ) and is called the radius vector, it is a directed distance, so it is (+ve) positive if measured along the arm of the angle $\boldsymbol{\theta}$ in the direction from $\boldsymbol{O}$ to $\boldsymbol{P}$ and
(-ve) negative if measured in the opposite direction. as shown in the figure (5.19).


Fig.(5.19)
5) The position of the point $\boldsymbol{P}$ is known when $(\boldsymbol{r})$ and $(\boldsymbol{\theta})$ are known.
6) The pair $(\boldsymbol{r}, \boldsymbol{\theta})$ is called a set of polar co-ordinates for the point $P$.
7) Every point in the plane has only one set of rectangular (Cartesian) co-ordinates, but it has infinitely many sets of polar co-ordinates i.e., the polar representation of a point $\boldsymbol{P}$ is not unique (only one), i.e., the same point $\boldsymbol{P}$ may be determined by different pairs of polar co-ordinates.
So that if $(\boldsymbol{r}, \boldsymbol{\theta})$ is one set of polar co-ordinates for $\boldsymbol{P}$ then other sets will be in one of the following forms:

$$
(r, \theta+2 n+\pi),(-r, \theta+(2 n+1) \pi) \text { where } \pi=180^{\circ}, n \text { is any }
$$ integer, as shown in the following figure (5.20).



Fig.(5.20)

## Example:

The point $\boldsymbol{P}$ in fig $(20, b)$ can be represented as $\left(\mathbf{2}, \frac{\pi}{3}\right)$ or $\left(\mathbf{2}, \frac{\pi}{3}+\mathbf{2 n} \boldsymbol{\pi}\right)$ or ( $-2, \frac{4 \pi}{3}+2 n \pi$ ).
8) Despite the unlimited number of sets of polar co-ordinates for any point $P$ other than the origin $\boldsymbol{O}$, we stress that $\boldsymbol{P}$ has only one set of polar co-ordinates ( $\mathbf{r}, \boldsymbol{\theta}$ ) such that:
$r>0(+\mathrm{ve})$ and $0 \leq \theta<2 \pi$.
9) The pole (origin) is assigned radius vector $(r=0)$ and arbitrary polar angle $(\theta)$ and is denoted by $\boldsymbol{O}(\mathbf{0}, \boldsymbol{\theta})$.
10) Although most maps use Cartesian co-ordinates, maps in polar coordinates do exist. In addition, the dance a bee performs to communicate the location of a source of food seems to be related to polar co-ordinates.
The orientation of its body locates the direction of the food, and the intensity of the dance indicates the distance of the source.
11) Every point in the plane has both rectangular (Cartesian) and polar co-ordinates in many applications, it is advantageous to use both coordinates, and to convert from each type of co-ordinates to the other by using the transformation formulas indicating the relationship between these two systems as shown in the following figure (5.21).

Now let $P$ be a point in the plane with Cartesian co-ordinates $(x, y)$ and polar co-ordinates $(r, \theta)$, then from the definition of the sine and the cosine we deduce the following formulas:
$\frac{x}{r}=\cos \theta \longrightarrow x=r \cos \theta$
$\frac{y}{r}=\sin \theta \longrightarrow y=r \sin \theta$
$\frac{y}{x}=\tan \theta \longrightarrow y=x \tan \theta$
notice that from Pythagorean theorem we have $\boldsymbol{x}^{2}+\boldsymbol{y}^{2}=\boldsymbol{r}^{2}$ also, from the above formulas we get

$$
\begin{aligned}
x^{2}+y^{2} & =r^{2} \cos 2 \theta+r^{2} \sin 2 \theta \\
& =r^{2}(\cos 2 \theta+\sin 2 \theta) \\
& =r^{2}(1) \\
& =r^{2}
\end{aligned}
$$



Fig.(5.21)

Then $x$ and $y$ co-ordinates are determined by $\boldsymbol{r}$ and $\boldsymbol{\theta}$ as follows:

$$
x=r \cos \theta \quad \text { and } \quad y=r \sin \theta
$$

And the polar co-ordinates $r$ and $\theta$ of P are determined by $x$ and $y$ as follows:

$$
\begin{gathered}
r^{2}=x^{2}+y^{2} \quad r= \pm \sqrt{x^{2}+y^{2}} \\
\sin \theta= \pm \frac{y}{r} \quad \cos \theta= \pm \frac{x}{r} \quad \tan \theta=\frac{y}{x}
\end{gathered}
$$

b) Polar co-ordinates in (space) three - dimensions the position of the point $P$ in space is identified in this system of co-ordinates in terms of its distance from the origin O and the direction angles of $O P$ as shown in the figure (5.22).


Fig.(5.22)

## Notes:

1) The point $P$ is denoted by $P(r, \alpha, \beta, \gamma)$
2) $\quad r$ is the distance.
3) $\alpha, \beta$ and $\gamma$ are the direction angles of $O P$.
4) The relations connecting the polar and Cartesian co-ordinates
$x=r \cos \alpha, y=r \cos \beta, z=r \cos \gamma$.
$\mathrm{r}= \pm \sqrt{x^{2}+y^{2}+z^{2}}$, are used to convert from one system to the other.

### 5.2.2 Cylindrical co-ordinate system:

The position of the point P in space is identified in this system of coordinates in terms of the polar co-ordinates of its projection upon the $X \boldsymbol{Y}-$ plane as shown in the figure (5.23).


Fig.(5.23)

## Notes:

1) The point $\boldsymbol{P}$ is denoted by $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$.
2) $(\mathbf{r}, \boldsymbol{\theta})$ are polar co-ordinates of the point $\mathbf{Q}(\boldsymbol{x}, \boldsymbol{y}, \mathbf{0})$ the projection of $\boldsymbol{P}$ upon XY-plane.
3) $\boldsymbol{Z}$ is the Cartesian $\boldsymbol{Z}$ co-ordinate of $\boldsymbol{P}$.
4) To convert (change) rectangular co-ordinates of $\boldsymbol{P}$.

To cylindrical co-ordinates use the formulas:
$x=r \cos \theta, \quad y=r \sin \theta, z=z$
$r= \pm \sqrt{x^{2}+y^{2}}$

### 5.2.3 Spherical co-ordinate system:

The position of the point $P$ in space is identified in this system of coordinates in terms of its distance $r$ from the origin and the polar angle $\theta$ of its projection $\mathrm{Q}(x, y, 0)$ upon XY-plane and the angle $\Phi$ between OP and the positive Z-axis as shown in the figure (5.24)


Fig.(5.24)

## Notes:

1) The point $P$ is denoted by $(r, \theta, \emptyset)$.
2) ( $r, \theta, \emptyset)$ are called the spherical co-ordinates of the point $P$.
3) $\quad R$ is called the radius vector or the radial distance of $P$ being equals the length of the line segment $\mathrm{O} p$ and it is taken to be $\mathrm{r} \geq 0$.
4) The angle $\theta$ is called the a zimuth of $P$ or the longitude, and the angle $\varnothing$ is called the co-latitude of $P$ and they are taken as: $0 \leq \theta \leq 180$ , $0 \leq \Phi \leq 360$.
5) To convert (change) rectangular (Cartesian) co-ordinates to spherical and vice versa use

$$
\begin{aligned}
& x=r \sin \emptyset \cdot \cos \theta, \quad y=r \sin \emptyset \cdot \sin \theta \\
& z=r \cos \emptyset \quad r= \pm \sqrt{x^{2}+y^{2}+z^{2}}
\end{aligned}
$$

### 5.3 The distance formula

The undirected distance between the two points $P_{1}$ and $P_{2}$ which is called simply. The distance from either of these points to the other is defined to be the length of the line segment $P_{1} P_{2}$ and is denoted by $\left|P_{1} P_{2}\right|$ or $d\left(P_{1} P_{2}\right)$.The analytic geometry provided us with a formula in an algebraic form to calculate this distance, this formula came in terms of the rectangular (Cartesian) co-ordinates of these points and differs according to the dimensions of the space of these points as shown in the figure (5.25):


### 5.3.1. Distance formula in one - dimension:

If $\boldsymbol{P}_{\mathbf{1}}$ and $\boldsymbol{P}_{\mathbf{2}}$ are two points on the same horizontal line, i.e., parallel to the $\mathbf{X}$-axis (or on the axis itself), and also if $\boldsymbol{P}_{\mathbf{1}}$ and $\boldsymbol{P}_{\mathbf{2}}$ on the same vertical line, i.e., parallel to the $\mathbf{Y}$-axis (or on the axis itself), as shown in the figure ( $5.26, a, b)$, then the distance between them is the distance between their $\boldsymbol{x}$ coordinates or $\boldsymbol{y}$-coordinates receptivity i.e.

$$
\left|P_{1}-P_{2}\right|=\left|x_{2}-x_{1}\right|=\left|y_{2}-y_{1}\right| .
$$



Distance $=\left|y_{2}-y_{1}\right|$
(a)

distance $=\left|x_{2}-x_{1}\right|$
(b)

Fig(5.26)

### 5.3.2 Distance formula in two - dimensions (in plane)

If $\boldsymbol{P}_{\mathbf{1}}\left(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{y}_{\mathbf{1}}\right)$ and $\boldsymbol{P}_{\mathbf{2}}\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}\right)$ are any two points in the rectangular (Cartesian) co-ordinate plane then the distance $\left|\boldsymbol{P}_{\mathbf{1}} \boldsymbol{P}_{\mathbf{2}}\right|$ between them is given by the formula:

$$
\left|P_{1} P_{2}\right|=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
$$

By applying the Pythagorean theorem to the right triangle in the figure (5.27) we can see that $d^{2}=\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}$


Fig.(5.27)

$$
\begin{gathered}
\therefore d=\left|P_{1} P_{2}\right|=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}} \text { is the required distance } \\
\text { formula in plane. }
\end{gathered}
$$

The above formula may be obtained in another manner as follows:
As shown in the figure (27) if $\theta$ is the angle of inclination of $P_{1} P_{2}$ to the x -axis then

$$
d \cos \theta=x_{2}-x_{1} \quad, \quad d \sin \theta=y_{2}-y_{1}
$$

$$
\begin{aligned}
\therefore d^{2} \cos ^{2} \theta+d^{2} \sin ^{2} \theta & =\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2} \\
\therefore d^{2}\left(\cos ^{2} \theta+\sin ^{2} \theta\right) & =\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2} \\
,\left(\cos ^{2} \theta+\sin ^{2} \theta\right) & =1 \\
\therefore \quad d^{2} \cdot 1 & =\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2} \\
\therefore d=\left|P_{1} P_{2}\right| & =\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
\end{aligned}
$$

Is the required distance formula in plane.

### 5.3.3 - Distance formula in three - dimensions (in space)

If $P_{1}\left(x_{1}, y_{1}, z_{1}\right)$ and $P_{2}\left(x_{2}, y_{2}, z_{2}\right)$ are any two points in space. then the distance $\left|P_{1} P_{2}\right|$ between them is given by the formula

$$
\left|P_{1} P_{2}\right|=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
$$

This formula is derived by the use of the relation between the length $d$ of a diagonal of a rectangular parallel piped and the lengths $(a, b, c)$ of its three edges, as shown in the figure (5.28), which states that:
$d^{2}=a^{2}+b^{2}+c^{2}$


Fig.(5.28)
This is done by constructing a rectangular parallel piped with faces parallel to the co-ordinate planes and with $P_{1}$ and $P_{2}$ as the ends of a diagonal as shown in the figure (5.29). The lengths of the edges of this parallel piped are:

$$
\left|x_{2}-x_{1}\right|,\left|y_{2}-y_{1}\right| \text { and }\left|z_{2}-z_{1}\right|
$$



Fig.(5.29)

$$
\begin{gathered}
\therefore d^{2}=\left|P_{1} P_{2}\right|^{2}=\left|x_{2}-x_{1}\right|^{2}+\left|y_{2}-y_{1}\right|^{2}+\left|z_{2}-z_{1}\right|^{2} \\
\text { Or }\left|P_{1} P_{2}\right|^{2}=\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}+\left(z_{2}-z_{1}\right)^{2} \\
\therefore\left|P_{1} P_{2}\right|=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}+\left(z_{2}-z_{1}\right)^{2}}
\end{gathered}
$$

Is the required distance formula in space.

## Notes:

1) Whenever we consider the distance formula between two points in the rectangular (Cartesian) co-ordinate system assume that the same unit of length is used on the axes.
2) In using the distance formula to find the distance between two points, either points may be designated by ( $x_{1}, y_{1}$ ) the other designated by ( $x_{2}, y_{2}$ ).
3) $\left|P_{1} P_{2}\right|=\left|P_{2} P_{1}\right|$ due to the fact that the distance between any two points is the length of the line segment connecting these two points, thus it is undirected and positive (+ ve ).
4) Observer that since $\left(x_{2}-x_{1}\right)^{2}=\left(x_{1}-x_{2}\right)^{2}$ and $\left(y_{2}-y_{1}\right)^{2}=\left(y_{2}-y_{1}\right)^{2}$ and $\left(z_{2}-z_{1}\right)^{2}=\left(z_{2}-z_{1}\right)^{2}$ we may write:
$\left|P_{1} P_{2}\right|=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}$
Or $\quad=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}$ notice that the same idea be applied to the three - dimensional case.
5) If $P_{1}$ coincides with $P_{2}$, the distance $\left|P_{1} P_{2}\right|=0$
6) Particular cases of the distance formula between two points are:
i) The distance of a point $P(x, y)$ from the origins
$|O P|=r=\sqrt{(x-0)^{2}+(y-0)^{2}}=\sqrt{x^{2}+y^{2}}$ as shown in the figure (5.30). $P(x, y)$


Fig.(5.30)
ii) If the two points are on the X -axis i.e., $P_{1}\left(x_{1}, 0\right), P_{2}\left(x_{2}, 0\right)$ then

$$
\left|P_{1} P_{2}\right|=\sqrt{\left(x_{2}-x_{1}\right)^{2}+(0-0)^{2}}=\sqrt{\left(x_{2}-x_{1}\right)^{2}}=\left|x_{2}-x_{1}\right|
$$

iii) If the two points are on the y-axis i.e. $P_{1}\left(0, y_{1}\right), P_{2}\left(0, y_{2}\right)$ then

$$
\begin{aligned}
\left|P_{1} P_{2}\right| & =\sqrt{(0-0)^{2}+\left(y_{2}-y_{1}\right)^{2}}=\sqrt{\left(y_{2}-y_{1}\right)^{2}} \\
& =\left|y_{2}-y_{1}\right|
\end{aligned}
$$

iv) If the two points are on the same horizontal line
i.e., $P_{1}\left(x_{1}, y_{1}\right), P_{2}\left(x_{2}, y_{2}\right)$ then

$$
\begin{aligned}
\left|P_{1} P_{2}\right| & =\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}=\sqrt{\left(x_{2}-x_{1}\right)^{2}+0} \\
& =\left|x_{2}-x_{1}\right|
\end{aligned}
$$

v) If the two points are on the same vertical line
i.e. $\quad P_{1}\left(x_{1}, y_{1}\right), P_{2}\left(x_{2}, y_{2}\right)$, similarly

$$
\left|P_{1} P_{2}\right|=\sqrt{\left(y_{2}-y_{1}\right)^{2}}=\left|y_{2}-y_{1}\right|
$$

7) The distance formula has some applications concerned with the definition of some geometric concepts and relationships such as:
i) The straight line is defined in such a way using the concept of distance between two points as:
" The straight line " is a set of points $P(x, y)$ such that the distance between any three points equals the sum of the distances between each two successive points, i.e. $d=d_{1}+d_{2}$, as shown in the figure (5.31).


Fig(5.31)

Three points $P_{1}, P_{2}$ and $P_{3}$ are collinear i.e., they lie on the same straight line if they satisfy the condition that the distance between two of them equals the sum of the distances between each two in some order.see figure (5.32)


Fig.(5.32)
Such that

$$
\left|P_{1} P_{3}\right|=\left|P_{1} P_{2}\right|+\left|P_{2} P_{3}\right|
$$

ii) The circle with canter $P_{0}\left(x_{0}, y_{0}\right)$ and radius $r$ is defined to be the set of points $P(x, y)$ in the same plane (see the figure (5.33)), such that:
$\left|P_{0} P\right|=r$. Applying the distance formula, we derive the following equation of circle:

$\left|P_{0} P_{1}\right|=r \longrightarrow \sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}=r$
Or $\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}=r^{2}$ is the required equation notice that this equation describes points on the circumference of the circle only.
iii) The disk with center $P_{0}\left(x_{0}, y_{0}\right)$ and radius r is defined to be the set of points $P(x, y)$ in the same plane (see the figure (5.34)), such that $\left|P_{0} P_{1}\right| \leq r$.


Fig.(5.34)

Applying the distance formula, we derive the following equation of disk:

$$
\left|P_{0} P\right|=r \longrightarrow \sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}} \leq r
$$

Or $\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2} \leq r^{2}$ is the required equation notice that this equation describes the set of points on the circumference and inside the circle.
iv) The sphere with center $P_{0}\left(x_{0}, y_{0}, z_{0}\right)$ and radius $r$ (see the figure (5.35)), is defined to be the set of points $P(x, y, z)$ in space such that $\left|P_{0} P\right|=r$
Applying the distance formula, we can derive the following equation of sphere.

$$
\begin{aligned}
\left|P_{0} P\right|=r & \Longrightarrow \sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}}=r \\
& \text { or }\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}=r^{2}
\end{aligned}
$$



This is the required equation of sphere.
Notice that this equation describes the set of points on the surface of the sphere only.
v) The ball with center $P_{0}\left(x_{0}, y_{0}, z_{0}\right)$ and radius $r$ (see the figure (5.36)), is defined to be the set of points $P(x, y, z)$ in space such that $\left|P_{0} P\right| \leq r$.
Applying the distance formula, we can derive the following equation of ball:


Fig.(5.36)

$$
\left|P_{0} P\right|=r \Longrightarrow \sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}} \leq r
$$

Which is equivalent to:

$$
\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2} \leq r^{2}
$$

This is the required equation of a ball.
Notice that this equation describes the set of points on the surface and inside the sphere.

## Examples:

1) Find the distance between each two points in the following:
i) $\quad P_{1}(3,0), P_{2}(5,0) \longrightarrow\left|P_{1} P_{2}\right|=|5-3|=2$
ii) $\quad P_{1}(0,-5), P_{2}(0,4) \Longrightarrow\left|P_{1} P_{2}\right|=|4-(-5)|=9$
iii) $\quad P_{1}(2,3), P_{2}(7,3) \longrightarrow\left|P_{1} P_{2}\right|=|7-2|=5$
iv) $\quad P_{1}(4,6), P_{2}(4,-9) \Longrightarrow\left|P_{1} P_{2}\right|=|-9-6|=15$
v) $\quad P_{1}(3,4), P_{2}(-6,1) \longrightarrow\left|P_{1} P_{2}\right|=\sqrt{(-6-3)^{2}+(1-4)^{2}}$ $=\sqrt{(-9)^{2}+(-3)^{2}}$

$$
\begin{aligned}
& =\sqrt{81+9} \\
& =\sqrt{90}=3 \sqrt{10})
\end{aligned}
$$

vi) $\quad P_{1}(-1,3,2), P_{2}(1,2,3)$

$$
\begin{aligned}
&\left|P_{1} P_{2}\right|=\sqrt{(1+1)^{2}+(2-3)^{2}+(3-2)^{2}} \\
&= \sqrt{(2)^{2}+(-1)^{2}+(1)^{2}} \\
&= \sqrt{4+1+1} \\
&=\sqrt{6}
\end{aligned}
$$

2) In the figure $(5.37)$, show that $(2,4),(3,5)$ and $(5,7)$ are collinear (i.e. lie on straight line).


Fig.(5.37)

## Solution:

$$
\begin{aligned}
d_{1}=\left|P_{1} P_{2}\right|= & \sqrt{(3-2)^{2}+(5-4)^{2}} \\
& =\sqrt{(1)^{2}+(1)^{2}}=\sqrt{2} \\
d_{2}=\left|P_{2} P_{3}\right| & =\sqrt{(5-3)^{2}+(7-5)^{2}} \\
& =\sqrt{(2)^{2}+(2)^{2}} \\
& =\sqrt{4+4}=\sqrt{8} \\
& =\sqrt{2 \times 4}=2 \sqrt{2} \\
d=\left|P_{1} P_{3}\right| & =\sqrt{(5-2)^{2}+(7-4)^{2}} \\
& =\sqrt{(3)^{2}+(3)^{2}}
\end{aligned}
$$

$$
\begin{aligned}
& =\sqrt{9+9}=\sqrt{18} \\
& =\sqrt{2 \times 9}=3 \sqrt{2}
\end{aligned}
$$

Notice that $d=d_{1}+d_{2}$ i.e. $3 \sqrt{2}=\sqrt{2}+2 \sqrt{2}$
$\therefore P_{1}, P_{2}, P_{3}$ lie on straight line.

### 5.4 Division formula:

Which is also called section formula, or ratio formula. It is a formula concerned with determining the co-ordinates of the point which divides the distance between two points in a given ratio. (see the figure (5.38)),

Now let $P(x, y)$ divides the distance between $P_{1}\left(x_{1}, y_{1}\right)$ to $P_{2}\left(x_{2}, y_{2}\right)$ in the ratio $\mathrm{L}: \mathrm{M}$


Fig.(5.38a)


Fig.(5.38b)

Notice that three are two types of division as shown in the figure $(5.38, \mathrm{a}, \mathrm{b})$.
One of them is internal in which the point $P(x, y)$ of division lies between $P_{1}$ and $P_{2}$
i.e., divides the distance from inside, i.e. $\frac{\left|P_{1} P\right|}{\left|P_{2} P\right|}=\frac{L}{M}$
or $\left|P_{1} P\right|:\left|P P_{2}\right|=L: M$
The other is external in which the point $P(x, y)$ lies on the extension of the line segment $P_{1} P_{2}$ but beyond $P_{1}$ or $P_{2}$, i.e., divides the distance from out side (see figure (5.39)),
i.e. $\quad \frac{\left|P_{1} P\right|}{\left|P_{2} P\right|}=\frac{L}{M}$ or $\left|P_{1} P\right|:\left|P_{2} P\right|=L: M$

As in the case of distance formula the ratio formula can be derived (obtained) by two manners as follows:
The first method states that as shown in the figure (5.39) from the similar triangles we can obtain that

$$
\frac{\left|P_{1} P\right|}{\left|P P_{2}\right|}=\frac{x-x_{1}}{x_{2}-x}=\frac{y-y_{1}}{y_{2}-y}=\frac{L}{M}
$$



Fig.(5.39)
$\therefore$ Thus, we can obtain the following formulas:
$\frac{x-x_{1}}{x_{2}-x}=\frac{L}{M}=$
(1) $\frac{y-y_{1}}{y_{2}-y}=\frac{L}{M}$
$\frac{x-x_{1}}{x_{2}-x}=\frac{L}{M}$
$M\left(x-x_{1}\right)=L\left(x_{2}-x\right)$
$M x-M x_{1}=L x_{2}-L x \Longrightarrow L x+M x=L x_{2}+M x_{1}$
solve for $x$
$x(L+M)=L x_{2}+M x_{1} \Longrightarrow x=\frac{L x_{2}+M x_{1}}{L+M}$
in the same way equation (2) gives $\frac{L y_{2}+M y_{1}}{L+M}$
these are the formulas for finding the co-ordinates of the division point internally
i.e., the internal division point $P(x, y)$ is:

$$
P\left(\frac{L x_{2}+M x_{1}}{L+M}, \frac{L y_{2}+M y_{1}}{L+M}\right) .
$$

The second method indicated that with referring to figure (5.40) we can write the $\cos \theta=\frac{x-x_{1}}{L}$ and also
$\cos \theta=\frac{x_{2}-x}{M}$
$\therefore \frac{x-x_{1}}{L}=\frac{x_{2}-x}{M} \Longrightarrow \frac{x-x_{1}}{x_{2}-x}=\frac{L}{M}$
And also $\frac{y-y_{1}}{y_{2}-y_{1}}=\frac{L}{M}$
As indicated previously these two equations give the following formulas:
$x=\frac{L x_{2}+M x_{1}}{L+M} \quad$ and $\quad y=\frac{L y_{2}+M y_{1}}{L+M}$
$P\left(\frac{L x_{2}+M x_{1}}{L+M}, \frac{L y_{2}+M y_{1}}{L+M}\right)$ is the point of division internally.

external division: we can see that

$$
\left|\mathrm{P}_{1} \mathrm{P}\right|:\left|\mathrm{P}_{2} \mathrm{P}\right|=\mathrm{L}: \mathrm{M} \longrightarrow \frac{\left|P_{1} P\right|}{\left|P_{1} P\right|}=\frac{L}{M}
$$

i.e. $\frac{x-x_{1}}{x-x_{2}}=\frac{y-y_{1}}{y_{2}-y}=\frac{L}{M}$ thus
we can obtain the following formulas:

$$
\begin{aligned}
& \frac{x-x_{1}}{x_{2}-x}=\frac{\mathrm{L}}{\mathrm{M}} \ldots \ldots . .(1), \frac{y-y_{1}}{y_{2}-y}=\frac{\mathrm{L}}{\mathrm{M}} \ldots \ldots . .(2) \\
& \therefore \frac{x-x_{1}}{x_{2}-x}=\frac{\mathrm{L}}{\mathrm{M}} \longrightarrow \mathrm{M}\left(x-x_{1}\right)=\mathrm{L}\left(x-x_{2}\right) \\
& \mathrm{M} x-\mathrm{M} x_{1}=\mathrm{L} x-\mathrm{L} x_{2} \longrightarrow \mathrm{~L} x_{2}-\mathrm{M} x_{1}=\mathrm{L} x-\mathrm{M} x
\end{aligned}
$$

solve for $x$ :

$$
\begin{gathered}
x(\mathrm{~L}-\mathrm{M})=\mathrm{L} x_{2}-\mathrm{M} x_{1} \longrightarrow x=\mathrm{L} x_{2}-\mathrm{M} x_{1} \\
\therefore x=\frac{\mathrm{L} x_{2}-\mathrm{M} x_{1}}{\mathrm{~L}-\mathrm{M}} \text { similarly } y=\frac{\mathrm{L} y_{2}-\mathrm{M} y}{\mathrm{~L}-\mathrm{M}}
\end{gathered}
$$

These are the formulas for finding the co-ordinates of the division point externally which is $P\left(\frac{\mathrm{~L} x_{2}-\mathrm{M} x_{1}}{\mathrm{~L}-\mathrm{M}}, \frac{\mathrm{L} y_{2}-\mathrm{M} y_{1}}{\mathrm{~L}-\mathrm{M}}\right)$

## Notes:

1) Notice that in the division formula the co-ordinates of $P_{1}$ multiplied by M which is next to $\mathrm{P}_{2}$ while the co-ordinates of $\mathrm{P}_{2}$ are multiplied by L which is next to $\mathrm{P}_{1}$.
2) $\quad \mathrm{IF} \mathrm{L}=\mathrm{M}$ then the point $\mathrm{P}(x, y)$ of division will be the mid-point and called the point of bisection.
3) The two points which divide the distance between two given points into three equal pars are called points or trisection.
4) The ratio formula in three - dimensions (space) is the same as that in two - dimensions (plane)

$$
\therefore x=\frac{\mathrm{L} x_{2}+\mathrm{M} x_{1}}{\mathrm{~L}+\mathrm{M}}, y=\frac{\mathrm{L} y_{2}+\mathrm{M} y_{1}}{\mathrm{~L}+\mathrm{M}}, z=\frac{\mathrm{L} z_{2}+\mathrm{M} z_{1}}{\mathrm{~L}+\mathrm{M}}
$$

5) If P is an internal division point and Q is an external division point then $\mathrm{P}_{1} \mathrm{PP}_{2} \mathrm{Q}$ are said to form a harmonic range.

Examples: find the point which divides internally the distance from $\mathrm{P}_{1}(-5,1)$ to $\mathrm{P}_{2}(5,6)$ in the ratio $3: 2$

As shown in the figure (5.41) this is an internal division then

$$
\left.\begin{array}{l}
x=\frac{2(-5)+3(5)}{3+2}=\frac{-10+15}{5}=\frac{5}{5}=1 \\
y=\frac{2(1)+3(6)}{3+2}=\frac{2+18}{5}=\frac{20}{5}=4
\end{array}\right\} \quad \Longrightarrow \mathrm{P}(1,4)
$$


2) In the figure (5.42) find the point which divides externally the line joining $P_{1}(-3,-7)$ and $P_{2}(-1,-4)$ in the ratio $4: 3$

$$
\begin{aligned}
& x=\frac{4(-1)-3(-3)}{4-3}=\frac{-4+9}{1}=5 \\
& y=\frac{4(-4)-3(-7)}{4-3}=\frac{-16+21}{1}=5
\end{aligned}
$$



Fig.(5.42)

### 5.5. The straight line in two -dimension ( in plane ):

The straight line in the plane is defined to be a set of points $\mathrm{P}(x, y)$ in the same plane satisfying some particular linear first degree equation in two
variables in the form $\mathrm{A} x+\mathrm{B} y+\mathrm{C}=0$, where $\mathrm{A}, \mathrm{B}$ and C are real numbers and $\mathrm{A}, \mathrm{B}$ are not bot zeros.

We indicate that if we are given an equation of the above form and asked to plot its graph, we will notice that the graph of this equation is a straight line this mean that A linear equation in two variables represents a straight line in plane, and conversely the graph of every straight line in plane can be represented by a linear equation. This indicates that we are used to translate from an algebraic from (equation) to a geometric form (graph). In this section we are going to do an opposite translation i.e., from geometric to algebraic, which suggests that we have to derive an algebraic equation to the line from given information about this line.

In discussing the conditions of determining and identifying the straight line in the plane it is discovered that a straight line in the plane is completely determined in terms of the knowledge of its direction and a point lies on it thus we will discuss the equation of line in terms of these condition as follows:

### 5.5.1. Slope of a line in two dimensions (in plane ):

a) The inclination of a line L in plane is defined as the smallest positive angle measured from the positive direction of the X -axis to the line.see figure (5.43).

b) The slope of the line L is defined as the tangent of the angle of inclination and written as $\mathrm{M}=\tan \theta$.
c) as shown in the figure (5.44) the slope of line L passing through two points $\mathrm{P}_{1}\left(x_{1}, y_{1}\right), \mathrm{P}_{2}\left(x_{2}, \mathrm{y}_{2}\right)$ is $M=\tan \theta=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$


Fig.(5.44)

## Notes:

1) The necessary and sufficient conditions for determining and locating a straight line in plane are the knowledge of its angle of inclination and a point on it.
2) The straight line is simply called line and denoted by the letter $L$.
3) The direction of the line in plane is determined by its angle of inclination which is denoted by the letter $\theta$, such that
$0^{\circ} \leq \theta<180^{\circ},\left(\theta \neq 90^{\circ}\right)$.
4) The slope of a line $L$ is also called the gradient or tangent and denoted by the letter M.
5) The slope of a line is a precise measure of its steepness.
6) Any line parallel to the X -axis (i.e. horizontal) and the X -axis itself has inclination $\theta=0^{\circ}$ and hence a slope $\mathrm{M}=0$.
And any line parallel to Y - axis (i.e., vertical) and the Y -axis itself has inclination $\theta=90^{\circ}$ but no slope, because

$$
\mathrm{M}=\tan 90^{\circ} \longrightarrow \mathrm{M}=\infty \text { i.e., } \mathrm{M} \text { is undefined. }
$$

7) The formula $M=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{\text { differnce of the ordinates }}{\text { differnce of the obscissas }}$

$$
=\frac{\text { differnce of } y \text { cordinates }}{\text { differnce of } x \text { cordinates }}
$$

Is known as the slope formula this formula is still holds regardless of the quadrant in which the points $P_{1}$, and $P_{2}$ lie and regardless of their order i.e. it is immaterial which point is regarded as $P_{1}$, and which as $P_{2}$, Thus the slope formula may be written as:
$\mathrm{M}=\frac{y_{1}-y_{2}}{x_{1}-x_{2}}$ or $\mathrm{M}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$ on the condition that the difference being taken in the same order, this indicates that
$\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{y_{1}-y_{2}}{x_{1}-x_{2}}=\frac{\text { differnce in } y \text { cordinates }}{\text { differnce in } x \text { cordinates }}$
8) The angle $\theta$ between two lies $L_{1}$ and $L_{2}$ having inclinations $\theta_{1}$ and $\theta_{2}$ and slopes $M_{1}$ and $M_{2}$ respectively as shown in the figure (5.45).


Fig.(5.45)

$$
\theta_{1}=\theta_{2}+\theta \text { then } \theta=\theta_{1}-\theta_{2}
$$

$$
\tan \theta=\tan \left(\theta_{1}-\theta_{2}\right)
$$

$$
= \pm \begin{gathered}
\tan \theta_{1}-\tan \theta_{2} \\
1+\tan \theta_{1} \cdot \tan \theta_{2}
\end{gathered}
$$

$= \pm \frac{M_{1}-M_{2}}{1+M_{1} M_{2}}$ notice that the $\pm$ sign is used to indicate that there are two angles one of them is acute and the other is its supplement.
9) Two non-vertical lines $L_{1}$ and $L_{2}$ are parallel if their slopes $M_{1}$ and $M_{2}$ are equal: i.e. $L_{1} / / L_{2} \longleftrightarrow M_{1}=M_{2}$.
From figure (5.46) it appears that if two line are parallel, they have the same inclination i.e. $\theta_{1}=\theta_{2}$


Fig.(5.46)
And hence the same slope i.e. $\theta_{1}=\tan \theta_{2}, \mathrm{M}_{1}=\mathrm{M}_{2}$
And vice versa notice that any two vertical lines are parallel and any two horizontal lines are parallel.such as in the figure (5.47).


Fig.(5.47)
10) Two non-vertical lines $\mathrm{L}_{1}$ and $\mathrm{L}_{2}$ are perpendicular (normal to each other) if their slopes $\mathrm{M}_{1}$ and $\mathrm{M}_{2}$ are negative reciprocals).
i.e. $\mathrm{L}_{1} \perp \mathrm{~L}_{2} \Longleftrightarrow M_{1}=-\frac{1}{\mathrm{M}_{2}}, M_{2}=-\frac{1}{\mathrm{M}_{1}}$ or $\mathrm{M}_{1} \cdot \mathrm{M}_{2}=-1$ from figure (5.48) it appears that if the two lines $\mathrm{L}_{1}$ and $\mathrm{L}_{2}$ are perpendicular and their angles of inclination are $\theta_{1}$ and $\theta_{2}$ respectively, then $\theta_{2}=\theta_{1}+90^{\circ}$ hence $\tan \theta=\tan \left(\theta_{1}+90^{\circ}\right)$

$$
\begin{aligned}
& =-\cot \theta_{1} \\
& =-\frac{1}{\tan \theta_{1}}
\end{aligned}
$$

$\therefore M_{2}=-\frac{1}{\mathrm{M}_{1}} \quad$ or $\quad \mathrm{M}_{1} \cdot \mathrm{M}_{2}=-1$.


Fig.(5.48)
Notice that this relationship does not apply to a vertical and horizontal line however they are perpendicular since the slope of the horizontal line is (0) and the vertical is $(\infty)$ i.e., undefined. The $\mathrm{M}_{1} \cdot \mathrm{M}_{2}=0 . \infty$ i.e. $\mathrm{M}_{1} \cdot \mathrm{M}_{2} \neq-1$. 11) Three points $P_{1}, P_{2}$ and $P_{3}$ are collinear if the slope of $P_{1} P_{2}=$ slope of $\mathrm{P}_{2} \mathrm{P}_{3}$ as shown in the figure (5.49).


If, $\mathrm{P}_{1}, \mathrm{P}_{2}$ and $\mathrm{P}_{3}$ are on a straight line i.e. collinear

Then $\mathrm{M}_{\mathrm{P}_{1} \mathrm{P}_{2}}=\mathrm{M}_{\mathrm{P}_{1} \mathrm{P}_{3}}=\mathrm{M}_{\mathrm{P}_{2} \mathrm{P}_{3}}$, if not then the three points form a triangle i.e. they are vertices of a triangle.
12) To identify the angle of inclination and to determine the slope of a line from its graph draw two coordinate axes at one of its points and measure the angle of inclination there and calculate $\mathrm{M}=\tan \theta$. figure (5.50) shows some cases:


Fig.(5.50)

## Examples:

1) Find the slope and the angle of inclination of the following lines through each of the following pairs of points:
i) $(-8,-4),(5,9)$
ii) $(10,-3),(14,-7)$
iii) $(11,6),(9,6)$
iv) $(5,4),(5,10)$

Solution:
Using formula $M=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\tan \theta$ or $M=\frac{y_{1}-y_{2}}{x_{1}-x_{2}}$
i) $\quad M=\frac{9+4}{5+8}=\frac{13}{13} \quad$ or $\quad M=\frac{-4-9}{-5-8}=\frac{-13}{-13}=1$
$\therefore \tan \theta=1 \Longrightarrow \theta=\tan ^{-1} 1 \Longrightarrow \theta=45^{\circ}$
ii) $\quad M=\frac{-7+3}{14-10}=\frac{-4}{4}$ or $M=\frac{-3+7}{10-14} \frac{4}{-4}=-1$
$\therefore \tan \theta=-1 \quad \Longrightarrow \theta=\tan ^{-1}-1 \quad \Longrightarrow \quad \theta=135^{\circ}$
iii) $\quad M=\frac{6-6}{9-11}=\frac{0}{-2}=0 \quad$ or $\quad M=\frac{6-6}{11-9}=\frac{0}{2}=0$
$\therefore \tan \theta=0 \Longrightarrow \theta=\tan ^{-1} 0 \Longrightarrow \theta=0^{\circ}$
$\theta=0^{\circ} \longrightarrow$ the line is horizontal
iv) $\quad M=\frac{10-4}{5-5}=\frac{6}{0} \quad$ or $M=\frac{4-10}{5-5}=\frac{-6}{0}=-\infty$

M is undefined then the line is vertical then $\theta=90^{\circ}$
2) If $\mathrm{P}_{1}(-1,3), \mathrm{P}_{2}(1,0), \mathrm{P}_{3}(4,2), \mathrm{P}_{4}(0,8)$

Prove that: i) $\mathrm{P}_{1} \mathrm{P}_{2} \perp \mathrm{P}_{2} \mathrm{P}_{3}$
ii) $\mathrm{P}_{1} \mathrm{P}_{2}| | \mathrm{P}_{3} \mathrm{P}_{4}$

## Solution:

i) $\quad \mathrm{M}_{1}$ of $=\frac{0-3}{1+1}=\frac{-3}{2}$
$\mathrm{M}_{2}$ of $P_{2} P_{3}=\frac{2-0}{4-1}=\frac{2}{3}$
$\mathrm{M}_{3}$ of $P_{3} P_{4}=\frac{8-2}{0-4}=\frac{6}{-4}=\frac{3}{-2}$
$\mathrm{M}_{1}=\mathrm{M}_{3} \longrightarrow \mathrm{P}_{1} \mathrm{P}_{2}| | \mathrm{P}_{3} \mathrm{P}_{4}$
$M_{1} \cdot M_{2}=\frac{-3}{2} \cdot \frac{2}{3}=-1 \quad \therefore \mathrm{P}_{1} \mathrm{P}_{2} \perp \mathrm{P}_{2} \mathrm{P}_{3}$.
3) If $P_{1}(5,7), P_{2}(-3,1), P_{3}(-7,-2)$. prove that $P_{1}, P_{2}$ and $P_{3}$ are collinear .
$\mathrm{M}_{1}$ of $P_{1} P_{2}=\frac{1-7}{-3-5}=\frac{3}{4} \quad, \mathrm{M}_{2}$ of $P_{2} P_{3}=\frac{-2-1}{-7+3}=\frac{3}{4}$
$\mathrm{M}_{1}=\mathrm{M}_{2} \longrightarrow \mathrm{P}_{1}, \mathrm{P}_{2}$ and $\mathrm{P}_{3}$ are collinear.
4) Find the acute angle between the two lines with slop $-\frac{1}{2}$ and $\frac{1}{3}$
$\therefore \tan \theta= \pm \frac{\mathrm{M}_{1}-\mathrm{M}_{2}}{1+\mathrm{M}_{1} \mathrm{M}_{2}}$
$\therefore \tan \theta= \pm \frac{-\frac{1}{2}-\frac{1}{3}}{1+\left(-\frac{1}{2} \cdot \frac{1}{3}\right)}= \pm \frac{-\frac{-3-2}{6}}{1-\left(\frac{1}{6}\right)}= \pm \frac{-\frac{-5}{6}}{\left(\frac{6-1}{6}\right)}= \pm \frac{-\frac{-5}{6}}{\left(\frac{5}{6}\right)}=$ $\frac{\frac{5}{6}}{\left(\frac{5}{6}\right)}=1$
$\therefore$ the acute angle is 45 .
5) Prove that the angle between any two parallel lines is zero and between perpendicular line is $90^{\circ}$.
i) If $L_{1} / / L_{2} \longrightarrow M_{1}=M_{2}$ let it be $M$ then

$$
\tan \theta=\frac{\mathrm{M}_{1}-\mathrm{M}_{2}}{1+\mathrm{M}_{1} \mathrm{M}_{2}}=\frac{\mathrm{M}-\mathrm{M}}{1+\mathrm{MM}}=\frac{0}{1+\mathrm{M}^{2}}=0 \quad \therefore \theta=0^{\circ}
$$

ii) If $L_{1} \perp \mathrm{~L}_{2}$ then $\mathrm{M}_{1} \cdot \mathrm{M}_{2}=-1$ then

$$
\begin{aligned}
& \tan \theta=\frac{\mathrm{M}_{1}-\mathrm{M}_{2}}{1+\mathrm{M}_{1} \mathrm{M}_{2}}=\frac{\mathrm{M}_{1}-\mathrm{M}_{2}}{1+(-1)}=\frac{\mathrm{M}_{1}-\mathrm{M}_{2}}{1-1}=\frac{\mathrm{M}_{1}-\mathrm{M}_{2}}{0}=\infty \text { (undefined) } \\
& \therefore \theta=90^{\circ}
\end{aligned}
$$

### 5.5.2 Equation of straight line in two -dimensions (in plane):

The graph of a line is a geometric picture and a semi-concrete model representing this figure, while the equation is an algebraic picture and an abstract model representing it.

There are several forms of the equation of a line in the plane, we will try to derive them following the whole to parts stepping principle in which we will select one of the forms called the point - slope form to be the main form i.e., the reference form i.e., the central form and derive the other forms with respect to it as parts of this whole form as follows:

1) Point - slope form:

The equation of the line through point $\mathrm{P}\left(x_{1}, y_{1}\right)$ whose slope is M is $y-y_{1}=\mathrm{M}\left(x-x_{1}\right)$ as shown in the figure (5.51).


Fig.(5.51)
if $\mathrm{P}(x, y)$ is any point (general point) on L then $M=\frac{y-y_{1}}{x-x 1}$
or $y-y_{1}=\mathrm{M}\left(x-x_{1}\right)$ which is called the point - slope form and we will take it as the main form to derive the other forms.
2) Two - point form:

The equation of the line through the two points $\mathrm{P}_{1}\left(x_{1}, y_{1}\right)$ and $\mathrm{P}_{2}\left(x_{2}, y_{2}\right)$ is

$$
\frac{y-y_{1}}{x-x_{1}}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

As shown in the figure (5.52)


Fig.(5.52)
The slope of ( L ) is $\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$

Applying the slope - point form we get
$y-y_{1}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\left(x-x_{1}\right)$
Simplifying the equation we get
$\left(y-y_{1}\right)\left(x_{2}-x_{1}\right)=\left(y_{2}-y_{1}\right)\left(x-x_{1}\right)$
And written as:
$\frac{y-y_{1}}{x-x_{1}}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$.
Which is called the two - points form.
3) slope - intercept form:

The equation of the line having slope, M and Y -intercept b is

$$
y=\mathrm{M} x+\mathrm{b}
$$

As shown in the figure (5.53) line L intersect $\mathrm{Y}-$ axis at $\mathrm{P}_{1}(0, b)$.
i.e. $p_{1}(0, b) \in L$


Fig.(5.53)
Applying the point-slope form we get
$y-\mathrm{b}=\mathrm{M}(x-0) \quad \Longrightarrow y-\mathrm{b}=\mathrm{M} x$
$\therefore y=\mathrm{M} x+\mathrm{b}$ which is called the slope- intercept form.
4) Two - intercepts form:

The equation of the line whose $x$ and $y$ intercepts are a and b respectively is $\frac{x}{\mathrm{a}}+\frac{y}{\mathrm{~b}}=1$.
As shown in the figure (5.54), L intersects the $x$ and $y$ axes at the points $\mathrm{P}_{1}(\mathrm{a}, 0)$ and $\mathrm{P}_{2}(0, \mathrm{~b})$ respectively.
The slope of L is $M=\frac{\mathrm{b}-\mathrm{O}}{\mathrm{O}-\mathrm{a}}=\frac{-\mathrm{b}}{\mathrm{a}}$


Fig.(5.54)

Applying the point - slope form we get:

$$
y-0=\frac{\mathrm{b}}{\mathrm{a}}(x-a) \quad \Longrightarrow y=-\frac{\mathrm{b}}{\mathrm{a}}(x-\mathrm{a})
$$

Simplifying the equation, we get:
$\mathrm{a} y=-\mathrm{b}(x-\mathrm{a}) \Longrightarrow \mathrm{a} y=-\mathrm{b} x+\mathrm{ab}$
$\therefore \mathrm{b} x+\mathrm{a} y=\mathrm{ab} \quad \longrightarrow \quad \frac{\mathrm{b} x}{\mathrm{ab}}+\frac{\mathrm{a} y}{\mathrm{ab}}=1$
$\therefore \frac{x}{\mathrm{a}}+\frac{y}{\mathrm{~b}}=1$ which is called two - intercepts form.
5) The general form:

Any equation of the line in plane in any form of the previous forms can be simplified and put in the form.
$\mathrm{AX}+\mathrm{BY}+\mathrm{C}=0$, where $\mathrm{A}, \mathrm{B}$ and C are real numbers, and $\mathrm{A}, \mathrm{B}$ are not both zero.
This equation is called the general form of the equation of a line in plane and it is a first-degree equation in two variables X and Y which is also called a linear equation being its graph is a straight line.

## Notes:

1) Every line in the co-ordinates plane is the graph of an equation of the first degree in two variables in the form $\mathrm{AX}+\mathrm{BY}+\mathrm{C}=0$ where $\mathrm{A}, \mathrm{B}$ and $\mathrm{C} \in \mathrm{R}$, such that A and B are not both zero the converse of this is that the graph of any equation of the first degree in two variables.
In the form $\mathrm{AX}+\mathrm{BY}+\mathrm{C}=0$ is a line. This equation is also called a linear equation.

This means that this equation is the algebraic analog (model) of the graph of a line in the plane and the graph is the geometric analog (model) of the equation of this line.
2) The graph of an equation is the set of all points whose co-ordinates satisfy this equation, conversely the equation of a graph is an algebraic equation such that any point on this graph satisfies this equation.
3) It appears that the two main problems of analytic geometry are drawing (plotting) of the graph of an equation and finding an equation of the graph.
4) The equation $A X+B Y+C=0$ is called the general first-degree equation in two variables or the general linear equation in two variables or the general form of equation of a line in the plane.
The word general is used because this form of equation includes all possible forms of the first-degree equations in two variables.
5) Referring to the general form of equation of line in the plane i.e.
6) $\mathbf{A} \mathbf{X}+\mathbf{B} \mathbf{Y}+\mathbf{C}=\mathbf{0}$ we can obtain the following special cases:
i) If $B \neq 0$ we can put the equation in the form:
$\mathrm{Y}=\mathrm{MX}+\mathrm{b}$ as $Y=-\frac{\mathrm{A}}{\mathrm{B}} X-\frac{\mathrm{C}}{\mathrm{B}} \quad \Longrightarrow \quad \mathrm{M}=-\frac{\mathrm{A}}{\mathrm{B}}$
ii) If $B=0$ then $A \neq 0$ then we can write the equation in the form of
$\mathrm{A} \mathrm{X}+\mathrm{C}=0$ or $\quad X=-\frac{\mathrm{C}}{\mathrm{A}}$ which is an equation of a vertical line (parallel to $\mathrm{Y}-$ axis ).
iii) If $\mathrm{A}=0$ then $\mathrm{B} \neq 0$ then we can write the equation in the form
$\mathrm{B} Y+\mathrm{C}=0$ or $Y=-\frac{\mathrm{C}}{\mathrm{B}}$ which is an equation of a horizontal line (parallel to X - axis).see the figure (5.55)


Fig.(5.55)

This indicates that the line $\mathrm{A} X+B Y+C=0$ is parallel to the $\mathrm{X}-$ axis if $\mathrm{A}=0$, and parallel to the $\mathrm{Y}-$ axis if $\mathrm{B}=0$ and passing through the origin (0) if $\mathrm{C}=0$.
7) In graphing an equation of a line in the plane it is sufficient to find two points $P_{1}\left(x_{1}, y_{1}\right)$ and $P_{2}\left(x_{2}, y_{2}\right)$ satisfying this equation, and plot these two points on a squared paper and draw the line through them, it will be the required graph of this line, notice that two point are enough (sufficient) since two points determine a line according to Euclidian geometry.
8) Referring to the slope - intercept form i.e. $y=\mathrm{m} x+\mathrm{b}$, we can obtain the following special cases:
i) The $y$ - intercept b may be positive (+ve) or negative ( -ve ) or zero according to the position of the cut point of the Y - axis as shown in the figure (5.56).

i) if $\mathrm{b}>0 \Longrightarrow$ the cut point is above the X - axis.
ii) if $\mathrm{b}<0$ the cut point is under (below) the X -axis.
iii) if $b=0$ then the line passes through the origin and the equation will be $: y=m x$.
iv) if $\mathrm{m}=1$ and $\mathrm{b}=0$ then the equation will be $y=x$ and the line passes through the origin and has an angle of inclination $\theta=45^{\circ}$, as shown in the figure (5.57). On this line the $x$ and $y$ coordinates of each point are equal.


Fig.(5.57)
v) if $\mathrm{m}=0$ then the equation will be $y=\mathrm{b}$ which is an equation of a line parallel to the X - axis, as shown in the figure (5.58).

On this line the $y$-coordinate of all points is fixed and equal $b$.


Fig.(5.58)
9) If a line $L$ is parallel to the $X$ - axis then it is called a horizontal line and has slope $\mathrm{m}=0$ and inclination $\theta=0^{\circ}$ and its equation is in the form $\boldsymbol{y}=\mathbf{b}$, and every point on it has the same $y$-coordinate $(=b)$.
10) If a line $\mathbf{L}$ is parallel to the $Y$ - axis then is called a vertical line and has inclination $\theta=90^{\circ}$ and no slope (undefined) and its equation is in the form $\boldsymbol{x}=\boldsymbol{a}$ and every point on it has the same $\boldsymbol{x}$-coordinate ( $=\mathrm{a}$ )
i) The $X$ - axis its self has slope $\boldsymbol{m}=\mathbf{0}$ and inclination $\theta=0^{\circ}$ and its equation is $\boldsymbol{y}=\mathbf{0}$
ii) The $\mathrm{Y}=$ axis itself has no slope and angle of inclination $\theta=90^{\circ}$ and its equation is $x=0$
11) To find the slope of a line from its equation put this equation in the slope - intercept form i.e. $\boldsymbol{y}=\boldsymbol{m} \boldsymbol{x}+\boldsymbol{b}$ then the coefficient of $\boldsymbol{x}$ which is m will be the required slope.
12) Notice that we can derive a short and quick formula for the slope of a line from its general equation as follows: put the general equation in the slope - intercept form i.e. if $B \neq 0$ then

$$
\begin{aligned}
& \mathrm{L}: \mathrm{AX}+\mathrm{B} \mathrm{Y}+\mathrm{C}=0 \Longrightarrow \mathrm{~B} \mathrm{Y}=-\mathrm{A} \mathrm{X}-\mathrm{C} \\
& \therefore Y=\frac{-\mathrm{A}}{\mathrm{~B}} X-\frac{-\mathrm{C}}{\mathrm{~B}} \Longrightarrow M=\frac{-\mathrm{A}}{\mathrm{~B}} \quad \therefore M=-\left(\frac{\mathrm{A}}{\mathrm{~B}}\right)
\end{aligned}
$$

$\therefore$ the slope of $\mathbf{L}$ is equal to minus the coefficient of x divided by coefficient of $y$, which is easy to remember if written as:
$M=-\left(\frac{\text { coefficient of } \mathrm{x}}{\text { coefficient of } \mathrm{y}}\right)$
13) The lines with equations $x=\mathrm{a}$ and $y=\mathrm{b}$ are perpendicular to each other since one of them is horizontal $(y=\mathrm{b})$ and the other is vertical $(x=a)$.
14) The angle between two lines (see the figure (5.59):
$\mathrm{L}_{1}: \mathrm{A}_{1} \mathrm{X}+\mathrm{B}_{1} \mathrm{Y}+\mathrm{C}_{1}=0 \quad$ and $\quad \mathrm{L}_{2}: \mathrm{A}_{2} \mathrm{X}+\mathrm{B}_{2} \mathrm{Y}+\mathrm{C}_{2}=0 \quad$ can be derived as :

$$
\begin{aligned}
& M_{1}=-\frac{\mathrm{A}_{1}}{\mathrm{~B}_{1}}, M_{2}=-\frac{\mathrm{A}_{2}}{\mathrm{~B}_{2}} \quad \text { and } \tan \theta= \pm \frac{\mathrm{M}_{1}-\mathrm{M}_{2}}{1+\mathrm{M}_{1} \mathrm{M}_{2}} \\
& \therefore \tan \theta= \pm \frac{-\frac{A_{1}}{B 1}+\frac{A_{2}}{B_{2}}}{1+\left(\frac{A_{1}}{B_{1}} \cdot \frac{A_{2}}{B_{2}}\right)} \quad \therefore \tan \theta= \pm \frac{\mathrm{A}_{1} \mathrm{~B}_{2}-\mathrm{A}_{2} \mathrm{~B}_{1}}{\mathrm{~A}_{1} \mathrm{~A}_{2}+\mathrm{B}_{1} \mathrm{~B}_{2}}
\end{aligned}
$$



Fig.(5.59)
15) The relationship between two lines in a plane
$\mathrm{L}_{1}: \mathrm{A}_{1} \mathrm{X}+\mathrm{B}_{1} \mathrm{Y}+\mathrm{C}_{1}=0$ and $\mathrm{L}_{2}: \mathrm{A}_{2} \mathrm{X}+\mathrm{B}_{2} \mathrm{Y}+\mathrm{C}_{2}=0$
Is either $L_{1}$ intersects $L_{2}$ in a point or are coincident or are parallel or are perpendicular. These relations can be deduced by putting the general from of each line in the slope - intercept form i.e. $L_{1}: y=m_{1} x+b_{1}$ and $\mathrm{L}_{2}: y=\mathrm{m}_{2} x+\mathrm{b}_{2}$, then by using the result of the previous notes we can reach to the following deductions. These deductions are called the conditions for parallelism and perpendicularity.
i) Two lines $L_{1}$ and $L_{2}$ are intersected at a point if they have precisely one point in common called the point of intersection which satisfies both equations of $L_{1}$ and $L_{2}$.
$\therefore \mathrm{M}_{1} \neq \mathrm{M}_{2}$ i.e. $\tan \theta= \pm \frac{\mathrm{A}_{1} \mathrm{~B}_{2}-\mathrm{A}_{2} \mathrm{~B}_{1}}{\mathrm{~A}_{1} \mathrm{~A}_{2}+\mathrm{B}_{1} \mathrm{~B}_{2}} \neq 0 \quad$ (where $\theta$ is the angle between $\mathrm{L}_{1}, \mathrm{~L}_{2}$ )

Then the lines intersect at a point to find the point of intersection of $L_{1}$ and $L_{2}$ solve the two equations of $L_{1}$ and $L_{2}$ simultaneously the solutions are the values of X and Y which corresponds to the co-ordinates of the common (intersection) point of these lines which will be

$$
P\left(\frac{\mathrm{~B}_{1} \mathrm{C}_{2}-\mathrm{B}_{2} \mathrm{C}_{1}}{\mathrm{~A}_{1} \mathrm{~B}_{2}-\mathrm{A}_{2} \mathrm{~B}_{1}}, \frac{\mathrm{C}_{1} \mathrm{~A}_{2}-\mathrm{C}_{2} \mathrm{~A}_{1}}{\mathrm{~A}_{1} \mathrm{~B}_{2}-\mathrm{A}_{2} \mathrm{~B}_{1}}\right) .
$$

ii) Two lines $L_{1}$ and $L_{2}$ are coincident if $M_{1}=M_{2}$ and $b_{1}=b_{2}$ i.e. $\frac{A_{1}}{B_{1}}=\frac{A_{2}}{B_{2}}$ and $\frac{C_{1}}{B_{1}}=\frac{C_{2}}{B_{2}}$
i.e. they have equal slopes and equal $y$-intercepts.
as shown in the figure (5.60), the coincident lines are actually the same line.


Fig.(5.60)
iii) Two lines $L_{1}$ and $L_{2}$ are parallel (//)

If $M_{1}=M_{2}$ and $b_{1} \neq b_{2}$ i.e. $\frac{A_{1}}{B_{1}}=\frac{A_{2}}{B_{2}}$ and $\frac{C_{1}}{B_{1}} \neq \frac{C_{2}}{B_{1}}$
i.e., they have equal slopes but different
$y=$ intercepts as shown in the figure (5.61).


Fig.(5.61)
iv) Two lines $L_{1}$ and $L_{2}$ are perpendicular ( $\perp$ )

If $M_{1} \cdot M_{2}=-1$ i.e. $A_{1} A_{2}+B_{1} B_{2}=0$ this is due to that
$\mathrm{L}_{1} \perp \mathrm{~L}_{2} \quad \Longrightarrow \mathrm{M}_{1} \cdot \mathrm{M}_{2}=-1$
$\therefore \frac{-\mathrm{A}_{1}}{\mathrm{~B}_{1}} \cdot-\frac{\mathrm{A}_{2}}{\mathrm{~B}_{2}}=-1 \quad \Longrightarrow \frac{\mathrm{~A}_{1}}{\mathrm{~B}_{1}} \cdot \frac{\mathrm{~A}_{2}}{\mathrm{~B}_{2}}=-1$
$\therefore \frac{\mathrm{A}_{1} \mathrm{~A}_{2}}{\mathrm{~B}_{1} \mathrm{~B}_{2}}=-1 \Longrightarrow \mathrm{~A}_{1} \mathrm{~A}_{2}=-\mathrm{B}_{1} \mathrm{~B}_{2}$
i.e. $\mathrm{A}_{1} \mathrm{~A}_{2}+\mathrm{B}_{1} \mathrm{~B}_{2}=0$.
16) The equation of the line passing through the point $P_{1}\left(x_{1}, y_{1}\right)$ and is parallel to (ii) perpendicular to the line
$\mathrm{L}: \mathrm{AX}+\mathrm{BY}+\mathrm{C}=0$ is
i) $\quad \mathrm{AX}+\mathrm{BY}=\mathrm{AX}+\mathrm{BY}_{1}$ is parallel to L
ii) $\quad \mathrm{BX}-\mathrm{AY}=\mathrm{BX}-\mathrm{A} \mathrm{Y}_{1}$ is perpendicular to L
17) The equation of a straight line is sometimes given in the form:
$\mathrm{X}=\mathrm{a}+\mathrm{bt}$ where $\mathrm{a}, \mathrm{b}, \mathrm{d} \in R$
$Y=c+d t$
Are real numbers (constants) and $t$ is a variable called parameter, these equations are then called parametric or freedom equations of the straight line, since

$$
\begin{aligned}
& t=\frac{x-\mathrm{a}}{\mathrm{~b}} \text { and } t=\frac{y-\mathrm{c}}{\mathrm{~d}} \text { then } \\
& \frac{x-\mathrm{a}}{\mathrm{~b}}=\frac{y-\mathrm{c}}{\mathrm{~d}} \Longrightarrow \mathrm{~b}(y-\mathrm{c})=\mathrm{d}(x-\mathrm{a})
\end{aligned}
$$

$\therefore y-\mathrm{c}=\frac{\mathrm{d}}{\mathrm{b}}(x-\mathrm{a})$ which is a line with slope $m=\frac{\mathrm{d}}{\mathrm{b}}$.
18) All of these forms of equations of line represent the same line i.e., in fact they are the same equation put in different shapes.
19) The general form of equation of a line is more "general " than the slope - intercept form ( $y=\mathrm{m} x+\mathrm{b}$ ) since it includes all possible lines where as the slope - intercept form does not include vertical lines, which are obtained from the general form $\mathrm{AX}+\mathrm{BY}+\mathrm{C}=0$ by setting $\mathrm{B}=0$ we get:

$$
\mathrm{AX}+\mathrm{C}=0 \Longleftrightarrow \mathrm{AX}=-\mathrm{C} \longrightarrow \mathrm{X}=-\frac{\mathrm{C}}{\mathrm{~A}} \text { which is a vertical line }
$$

20) The slope - point form $y-y_{1}=\mathrm{M}\left(x-x_{1}\right)$ is considered here as the main (central), essential or the whole form in deriving the various forms of line equations in plane and we suggest here that we are going to use this form always to find the equation of a line in any situation, notice that what we need to use this form is the knowledge of the slope $M$ and a point $\mathrm{P}_{1}\left(x_{1}, y_{1}\right)$ on this line.
21) Finally we mention that any of these forms of equation may be used for a given line depending on convenience, and we suggest here to use the slope - point formula $\mathrm{y}-y_{1}=\mathrm{M}\left(x-x_{1}\right)$ and then use any other convenient form and compare the results.

## Examples:

1) Find an equation of the line passing through $\mathrm{P}(-3,5)$ with slope $-\frac{3}{4}$. Solution: using the slope - point form i.e. $y-y_{1}=\mathrm{M}\left(x-x_{1}\right)$ we obtain :
$y-5=-\frac{3}{4}[x-(-3)]$, simplifying we get
$4 y-20=-3 x-9$ or $3 x+4 y-11=0$.
2) Find an equation of the line through the points $P_{1}(4,6)$ and $P_{2}(-1,3)$.

Solution: find the slope by the formula, $\mathrm{M}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$
$\mathrm{M}=\frac{3-6}{-1-4}=\frac{-3}{-5} \quad \therefore M=\frac{3}{5}$

- Using slope - point form i.e. $y-y_{1}=\mathrm{M}\left(x-x_{1}\right)$ we obtain
$y-6=\frac{3}{5}(x-4)$, simplifying we get
$3 x-5 y+18=0$.
- We can use the two - point form Directly
i.e. $\frac{y-y_{1}}{x-x_{1}}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}} \longrightarrow \frac{y-6}{x-4}=\frac{3-6}{-1-4}$
$\therefore \frac{y-6}{x-4}=\frac{3}{5} \Longrightarrow(y-6)=\frac{3}{5}(\mathrm{x}-4) \Longrightarrow 3 x-5 y+18=0$
- We can also use the slope - intercept form i.e. $y=\mathrm{m} x+\mathrm{b}$ we have $\mathrm{M}=\frac{3}{5}$.
$\therefore y=\frac{3}{5} x+\mathrm{b} \longrightarrow 3 x-5 y+5 \mathrm{~b}=0$, to find b we substitute in this equation with one of the two-point $\mathrm{P}_{1}$ or $\mathrm{P}_{2}$, we obtain $3(4)-5(6)-\mathrm{b}=0$. Then:

$$
12-30+5 \mathrm{~b}=0 \longrightarrow-18+5 \mathrm{~b}=0 \quad \therefore \quad b=\frac{18}{5}
$$

$\therefore$ the required equation is

$$
3 x-5 y+18=0
$$

3) Find an equation of the line passing through the point $\mathrm{P}(2,3)$ and parallel to the line $\mathbf{2 x}-\mathbf{3 y}-\mathbf{1}=\mathbf{0}$., as shown in the figure (5.62).

## Solution:

$$
\mathrm{L}_{1} / / \mathrm{L}_{2} \longrightarrow \mathrm{M}_{1}=\mathrm{M}_{2}
$$

To find M1, put the equation of $\mathbf{L} 1$ in the slope - intercept form i.e. $y=\mathrm{m} x+\mathrm{b}$, as follows :


Fig.(5.62)
$2 x-3 y-1=0 \Longrightarrow-3 y=-2 x+1$. Then
$y=\frac{-2}{-3} x-\frac{1}{3} \quad \longrightarrow \quad M_{1}=\frac{2}{3}$.
To find the slope $\mathrm{M}_{1}$ of the line $\mathrm{L}_{1}$ you may use the short formula:
$\mathrm{M}=-\left(\frac{\mathrm{A}}{\mathrm{B}}\right)$ then $M_{1}=-\left(\frac{2}{-3}\right) \Longrightarrow M_{1}=\frac{2}{3}$

- $\quad \mathrm{M}_{1}=\mathrm{M}_{2}=\frac{2}{3}$. To find the equation of $\mathrm{L}_{2}$ we may use the slope point form i.e. $y-y_{1}=\mathrm{M}\left(x-x_{1}\right)$ to obtain
$y-3=\frac{2}{3}(x-2)$, simplifying we get
$3 y-9=2 x-4 \quad \longrightarrow 2 x-3 y-4+9=0$
$\therefore$ the required equation is $2 x-3 y+5=0$
- We can also use the slope - intercept form i.e., $y=\mathrm{M} x+\mathrm{b}$ as follows:
We have $\mathrm{M}=\frac{2}{3} x+\mathrm{b}$, to find b use the point $\mathrm{P}(2,3)$ then
$3=\frac{2}{3}(2)+b \Longrightarrow 9=4+3 b$
$3 \mathrm{~b}=9-4 \quad \Longrightarrow \quad 3 \mathrm{~b}=5 \quad \Longrightarrow \mathrm{~b}=\frac{5}{3}$
$\therefore$ the required equation is $\mathrm{y}=\frac{2}{3} x+\frac{5}{3}$
$\therefore 2 x-3 y-5=0$.
- We can also use the equation which shows the relation between two parallel lines as $\mathrm{AX}+\mathrm{BY}=\mathrm{AX}+\mathrm{BY}_{1}$ for the line through $\mathrm{P}\left(x_{1}, y_{1}\right)$ and parallel to the given line .
$\therefore$ the required equation will be as follows

$$
\begin{aligned}
2 \mathrm{x}-3 \mathrm{y} & =2(2)+(-3)(3) \\
& =4-9 \\
& =-5 \Longrightarrow 2 x-3 y+5=0 .
\end{aligned}
$$

4) Find an equation of the line passing through the point $\mathrm{P}(-2,3)$ and perpendicular to the line $\mathbf{2 x}-\mathbf{3 y}+\mathbf{6}=\mathbf{0}$. as shown in the figure (5.63).

## Solution:

$$
\mathrm{L}_{1} \perp \mathrm{~L}_{2} \quad \Longrightarrow \quad \mathrm{M}_{1} \cdot \mathrm{M}_{2}=-1 \quad \therefore \mathrm{M}_{2}=-\frac{1}{\mathrm{M}_{1}}
$$



To find $M_{1}$ put the equation of $L_{1}$ in the slope - intercept form i.e. $y=m x+$ b as follows:
$2 x-3 y+6=0 \Longrightarrow-3 y=-2 x-6$ then
$y=\frac{-2}{-3} x+\frac{-6}{-3} \quad \Longrightarrow y=\frac{2}{3} x+2$
$\therefore M_{1}=\frac{2}{3}$ then $M_{2}=-\frac{3}{2}$.

- To find $\mathrm{M}_{1}$ of $\mathrm{L}_{1}$ we may use the short formula $M=-\left(\frac{\mathrm{A}}{\mathrm{B}}\right)$
$\therefore M_{1}=-\left(\frac{2}{3}\right) \Longrightarrow M_{1}=\frac{2}{3} \Longrightarrow M_{2}=-\frac{3}{2}$
- To find the equation of $L_{2}$ we may use the slope - point form i.e. $y-y_{1}=\mathrm{M}\left(x-x_{1}\right)$ to obtain $y-3=-\frac{3}{2}(x+2)$ simplifying to obtain $2 y-6=-3 x-6$

$$
\therefore 3 x+2 y=0 .
$$

* We can also use the slope - intercept form
i.e. $y=m x+b$ as follows
we have $M=\frac{-3}{2}$ Then the required equation is:
$y=\frac{-3}{2} x+b$, to find $b$ use the point $P(-2,3)$ then
$3=\frac{-3}{2}(-2)+\mathrm{b} \longrightarrow 3=3+b \quad b=0$
$\therefore$ the required equation of $L_{2}$ is $3 x+2 y=0$.
- We can also use the equation which shows the relation between two perpendicular lines as:
$\mathrm{BX}-\mathrm{AY}=\mathrm{BX}_{1}-\mathrm{AY}_{1}$, for the line through $\mathrm{P}\left(x_{1}, \mathrm{y}_{1}\right)$ and perpendicular to the line. $\mathrm{AX}+\mathrm{BY}+\mathrm{C}=0$
$\therefore$ the required equation will be as follows:

$$
\begin{aligned}
& -3 x-2 y=-3(-2)-2(3) \\
& \therefore-3 x-2 y=6-6 \Longrightarrow 3 x+2 y=0 .
\end{aligned}
$$

5) find the point of intersection of the two lines
$\mathrm{L}_{1}: 3 x-y-1=0$
$\mathrm{L}_{2}: x+2 y-5=0$

- We may solve the two equations simultaneously:

By multiplying equation (i) by 2 and add as follows:
$6 x-2 y-2=0$
$x+2 y-5=0$
$7 x \quad-7=0 \quad 7 x=7 \quad \therefore x=1$
Substitute in equation (1) to obtain:

$$
\begin{aligned}
3(1)-y-1=0 \Longrightarrow & -y=1-3 \\
= & -2 \longrightarrow y=2
\end{aligned}
$$

$\therefore$ the required point of interceding is $\mathrm{P}(1,2)$.

- We may use the formula $\mathrm{P}\left(\frac{\mathrm{B}_{1} \mathrm{C}_{2}-\mathrm{B}_{2} \mathrm{C}_{1}}{\mathrm{~A}_{1} \mathrm{~B}_{2}-\mathrm{A}_{2} \mathrm{~B}_{1}}, \frac{\mathrm{C}_{1} \mathrm{~A}_{2}-\mathrm{C}_{2} \mathrm{~A}_{1}}{\mathrm{~A}_{1} \mathrm{~B}_{2}-\mathrm{A}_{2} \mathrm{~B}_{1}}\right)$
$\therefore \mathrm{A}_{1}=3, \mathrm{~B}_{1}=-1, \mathrm{C}_{1}=-1, \mathrm{~A}_{2}=1, \mathrm{~B}_{2}=2, \mathrm{C}_{2}=-5$
$\therefore \mathrm{P}\left(\frac{5+2}{6+1}, \frac{-1+15}{6+1}\right) \therefore \mathrm{P}\left(\frac{7}{7}, \frac{14}{7}\right)$
$\therefore$ the required point of intersection is $\mathrm{P}(1,2)$
$6)$ Find the angle between the two lines.

$$
\mathrm{L}_{1}: 2 x+3 y-1=0 \text { and } \mathrm{L}_{2}: x-2 y+3=0
$$

Solution: use the formula $\tan \theta= \pm \frac{A_{1} B_{2}-A_{2} B_{1}}{A_{1} A_{2}+B_{1} B_{2}}$
$\mathrm{A}_{1}=2, \mathrm{~B}_{1}=3, \mathrm{C}_{1}=-1, \mathrm{~A}_{2}=1, \mathrm{~B}_{2}=-2, \mathrm{C}_{2}=-3$
$\therefore \tan \theta= \pm \frac{-6-3}{2-6}= \pm \frac{-9}{-4}= \pm \frac{9}{4}$
$\therefore \theta=\tan ^{-1} \frac{9}{4}$ and $\tan ^{-1}-\frac{9}{4}$
7)Test each of the following pairs of lines for parallelism and perpendicularity and intersection by putting each equation in the slope intercept form
i.e. $y=\mathrm{m} x+\mathrm{b}$ and compare the slopes:
i) $\mathrm{L}_{1}: x-2 y+3=0, \mathrm{~L}_{2}: 3 x-6 y+9=0$

$$
\begin{gathered}
\therefore \mathrm{L}_{1}:-2 y=-x-3, \mathrm{~L}_{2}:-6 y=-3 x-9 \\
\therefore \mathrm{~L}_{1}: y=\frac{1}{2} x+\frac{3}{2}, \mathrm{~L}_{2}:-y=\frac{1}{2} x+\frac{3}{2} \\
\therefore \mathrm{M}_{1}=\frac{1}{2}, \mathrm{~b}_{1}=\frac{3}{2}, \mathrm{M}_{2}=\frac{1}{2}, \mathrm{~b}_{2}=\frac{3}{2} \\
\mathrm{M}_{1}=\mathrm{M}_{2} \text { and } \mathrm{b}_{1}=\mathrm{b}_{2} \Longrightarrow \mathrm{~L}_{1} \text { and } \mathrm{L}_{2} \text { are coincident. }
\end{gathered}
$$

$$
\begin{aligned}
& \text { ii) } \mathrm{L}_{1}: 4 x+3 y-1=0, \mathrm{~L}_{2}: 8 x+6 y-3=0 \\
& \therefore \mathrm{~L}_{1}: 3 y=-4 x+1, \therefore \mathrm{~L}_{2}: 6 y=-8 x+3 \\
& \therefore \mathrm{~L}_{1}: y=\frac{-4}{3} x+\frac{1}{3}, \therefore \mathrm{~L}_{2}: y=\frac{-4}{3} x+\frac{1}{2} \\
& \therefore \mathrm{M}_{1}=\frac{-4}{3}, \mathrm{~b}_{1}=\frac{1}{3}, \mathrm{M}_{2}=\frac{-4}{3}, \mathrm{~b}_{2}=\frac{1}{2} \\
& \quad \mathrm{M}_{1}=\mathrm{M}_{2}=\frac{-4}{3} \text { and } \mathrm{b}_{1} \neq \mathrm{b}_{2}
\end{aligned}
$$

$\therefore \mathrm{L}_{1}$ and $\mathrm{L}_{2}$ are parallel
iii) $\quad \mathrm{L}_{1}: 2 x-3 y+5=0, \mathrm{~L}_{2}: 3 x+2 y-4=0$
$\therefore \mathrm{L}_{1}:-3 y=-2 x-5, \therefore \mathrm{~L}_{2}: 2 y=-3 x+4$
$\therefore \mathrm{L}_{1}: y=\frac{2}{3} x+\frac{5}{3}, \therefore \mathrm{~L}_{2}: y=\frac{-3}{2} x+2$
$\therefore \mathrm{M}_{1}=\frac{2}{3}, \mathrm{~b}_{1}=\frac{5}{3}, \mathrm{M}_{2}=\frac{-3}{2}, \mathrm{~b}_{2}=2$
$M_{1} \cdot M_{2}=\frac{2}{3} \cdot \frac{-3}{2}=-1$
$\therefore \mathrm{L}_{1}$ and $\mathrm{L}_{2}$ are perpendicular

## Unit. 6 Vector concepts

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## Unit. 6 Vector concepts

## Introduction

Vector mathematics is one of the most important topics in modern mathematics. the concept of vector appears as a result of representing some mechanical and physical quantities graphically.

Many applications of mathematics make use of the idea of vector and many abstract mathematical theories involve the concept of vector space, this indicates that the vector mathematics evolved in the last century is a response to the demand for the newly emerging physical sciences, which shows that. Vectors are important in pure and applied mathematics, they furnish a mean of studying geometry, algebra, trigonometry and mechanics, and the relations among them, vectors introduce a sufficient degree of simplification in to the study of these subjects, vectors appear constantly in the study of motion in space and describing force, and very useful in the areas of geometrical and physical applications.

Vector mathematics is divided into a number of integrated branches given names according to the area of application in which vectors are used such as:

- Vector algebra which is concerned with the algebraic operations on vectors and dealing with the symbolic methods in certain parts of geometry and mechanics.
- Vector geometry which is concerned with the treatment of plane and solid and analytic geometry and trigonometry in an easy way, in which formulae are easily recognizable as a special cases of vector equations, however the real power of vectors becomes clear in three-dimensions where we deal with solid analytic geometry and spherical trigonometry with vector methods which gives these subjects a more easy and clear form than the traditional approach.
- Vector analysis: which deals with the study of mechanics and physics with vector methods.
Vector analysis provides a concise notation for mathematics formulations of mechanical, physical and trigonometrically ideas. The above points indicate
that vector analysis is the study of the quantities that require a direction as well as a magnitude for their description.


## 6.1 - Vector definitions:

The vector concept is given several definitions according to the area of application in which it appears such as:

## Physical definition of Vector :

A vector is defined Physically as a quantity having both magnitude and direction.
Other references indicated that a non - zero vector is a combination of three things:
i) A magnitude
ii) A direction in space
iii) A sense (making more precise the idea of direction).

## Notes:

1) This definition is used more in the physics and mechanics fields.
2) Quantities that require magnitude as well as direction for their description are called vector quantities such as: velocity, acceleration, force, displacement current, and magnetic field.
3) Quantities that possess only magnitude are referred to as scalars they assume real values in their description such as: distance, temperature, speed, mass, length, time and any read number.
4) Graphically a vector is represented by an arrow (see the figure (6.1)) with the tip of the arrow designating.


Fig.(6.1)

Its direction and sense, the tail of the Arrow representing its starting point and the length of the arrow from tail to tip being proportion to the magnitude of the vector, the tail of the arrow is called the origin or initial point of the vector, and the tip (head) of the arrow is called the terminal point of the vector.
5) Analytically a vector is represented by a letter with an arrow over it as: $\overrightarrow{\boldsymbol{A}}$ and its magnitude is denoted by $|\overrightarrow{\boldsymbol{A}}|$
6) The vector with initial point $A$ and terminal (end point) point $B$ ( as shown in the figure (6.1)), will be denoted $\overrightarrow{A B}$.


Fig.(6.2)
7) The vector $\overrightarrow{A B}$ and $\overrightarrow{B A}$ have the same magnitude but opposite direction and opposite sense. (see the figure (6.3)).


Fig.(6.3)
8) The direction of a vector $\overrightarrow{\boldsymbol{A}}$ in two - dimensions is determined by the smallest positive angles $\propto$ and $\beta$ that $\overrightarrow{\boldsymbol{A}}$ makes with positive X and y - axis where:
$0 \leq \propto, \beta, \leq 180$ are called direction angles, and $\cos \alpha$, $\operatorname{Cos} \beta$ are called direction cosines of $\overrightarrow{\boldsymbol{A}}$ and $\cos ^{2} \alpha+\cos ^{2} \beta=1$.
Figures.(6.4) show the various possibilities for $\propto$ of $\beta$.

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أ. عبير خليل صليبي





Fig.(6.4)

In three - dimensions (space) the direction angles $\propto, \beta$ and $\gamma$ are defined to be the smallest positive angles that $\overrightarrow{\boldsymbol{A}}$ makes with the positive $X, Y$ and $Z$ - axes respectively. (see the figure (6.5)).


Notice that $0 \leq \propto, \beta, \gamma \leq 180$ and the angles $\propto, \beta$ and $\gamma$ are not usually in any of the co-ordinate planes. and that $\cos \alpha, \cos \beta$ and $\cos \gamma$ are called direction cosines of $\overrightarrow{\boldsymbol{A}}$, and also that
$\cos 2 \alpha+\cos 2 \beta+\cos 2 \gamma=1$.
9) The zero vector $\overrightarrow{\mathbf{0}}$ has neither direction angles nor direction cosines. its direction is not specified and its magnitude is zero i.e. $|\overrightarrow{0}|=0$.
10) Two vectors $\overrightarrow{\boldsymbol{A}}$ and $\overrightarrow{\boldsymbol{B}}$ are equal if they have the same magnitude and direction and sense, this indicates that every vector has infinitely many representative i.e. if $\overrightarrow{\boldsymbol{A}}$ is anon zero vector then any other vector parallel to $\overrightarrow{\boldsymbol{A}}$ with the same magnitude and sense is a representative to $\overrightarrow{\boldsymbol{A}}$ i.e. equal to $\vec{A}$.
11) Vectors $\overrightarrow{\boldsymbol{A}}$ and ( $-\overrightarrow{\boldsymbol{A}}$ ) have the same magnitude but opposite sense as shown in the figure. (6.6)


Fig.(6.6)
i.e. $\vec{A} / /-\vec{A}$ and $|\vec{A}|=|-\vec{A}|$ but $\vec{A} \neq-\vec{A}$

The vector $-\overrightarrow{\boldsymbol{A}}$ is called the negative of $\overrightarrow{\boldsymbol{A}}$.

### 6.1.1Geometric definition:

A vector is defined geometrically as an equivalence class of directed line segments, representing a translation, each line - segment being a representative of the vector.

## Notes:

1) The vector is represented graphically ( geometrically ) by a directed line - segment,
Thus; if A and $\mathbf{B}$ are two points, then the line - segment joining A and B represents a vector whose direction is that of the line - segment AB (see the figure (6.7), and whose sense is the direction of travel from A to B and magnitude is the distance between $A$ and $B$, this vector is denoted by:
$A B$ and $|A B|=d(A, B)$.


Fig.(6.7)
2) If points $A$ and $B$ are coincident i.e., they are the same point $C$ then the line - segment CC is called " degenerate " line - segment which is taken to represent the zero vector i.e. $\vec{C} \vec{C}=0$.

### 6.1.2.Algebraic definition:

The plane vector is defined algebraically as an ordered pair $[x, y]$ of real numbers called its components.

## Notes:

1) The first number $x$ is called the $x$-component of the vector, and the second number $y$ is called the y-component of the vector, and written as $\vec{A}=[x, y]$.
2) Similarly, the three - dimensional vector is defined as an ordered triple $[x, y, z]$ of real numbers called its components and written as $\vec{A}=[x, y, z]$.
3) In general, the n - dimensional vector is defined as an ordered n tuple of real numbers and written as $A=\left[a_{1}, a_{2}, a_{31}, \ldots, a_{n}\right]$
4) If A is a point and O is the origin, then the vector $\overrightarrow{O A}$ is called the position vector of the point A and written in a shortened form $\vec{a}$ the separate components of $\vec{a}$ are termed the coordinates of the point A i.e. if $\mathrm{A}\left(x_{1}, y_{1}\right)$ is the point then its position vector is $\underline{\vec{a}}=\left[x_{1}, y_{1}\right]$. As shown in the figure (6.8).


Fig.(6.8)
5) The magnitude or length or norm of the vector $\mathrm{A}=[x, y]$ is $|\vec{A}|=\sqrt{x^{2}+y^{2}} \mid$. similarly if $\mathrm{A}=[x, y, z]$ then

$$
(|\vec{A}|)=\sqrt{x^{2}+y^{2}+z^{2}}
$$

6) The vector $\overrightarrow{0}=[0,0]$ is called the zero or null vector which $|\overrightarrow{0}|=0$.

Also $\overrightarrow{0}=[0,0,0]$ is the zero vector in space.
7) Two vector $\vec{A}=\left[x_{1}, y_{1}\right]$ and $\vec{B}=\left[x_{2}, y_{2}\right]$ are equal i.e. $\vec{A}=\vec{B} \Longrightarrow\left[\mathrm{x}_{1}, \mathrm{y}_{1}\right]=\left[x_{2}, y_{2}\right]$ i.e. $x_{1}=x_{2}$ and $y_{1}=y_{2}$.
8) The vector is identified and determined and located and represented with respect to its initial point considering it as an origin point in plane this is done by drawing two perpendicular axes intersecting at this point.

- Geometrically of the vector can be represented in a polar form by considering the polar co-ordinates of its terminal (tip) point, and the vector is written in this form as $\vec{A}=\left[\mathrm{r}, \theta^{0}\right]$.(see the figure. (6.9)).

Fig.(6.9)


- Algebraically, the vector $\vec{A}$ can be represented by considering the Cartesian co-ordinates of its terminal (tip) point using a graph paper the vector can be written as $\vec{A}=\left[x_{1}, y_{1}\right]$.
The initial (tail) point considered as the origin. (see the figure. (6.10))


Fig.(6.10)

### 6.2.Vector algebra.

The algebra of vectors involves rules for combining vectors in various ways this branch of vector mathematics deals with performing of operations of addition, subtraction, and multiplication on vectors with suitable definition for each operation as follows:

### 6.2.1.scalar - vector multiplication:

if $\vec{A}=\left[x_{1}, y_{1}\right]$ is a plane vector and $\mathbf{k}$ is an arbitrary real number then $\mathbf{k} \vec{A}$ or $\quad \vec{A} \mathbf{k}$ has been defined algebraically to be the vector $\mathbf{k} \vec{A}=\left[\mathrm{k} x_{1}, \mathrm{k} y_{1}\right]$. in three - dimensions

$$
\mathrm{k} \vec{A}=\left[\mathrm{k} x_{1}, \mathrm{k} y_{1}, \mathrm{k} z_{1}\right] .
$$

## Notes:

1) $\quad|k \vec{A}|=|k||\vec{A}|$ which means that the magnitude (length) of the vector $\mathrm{k} \vec{A}$ is $|K|$ times the magnitude (length) of $\vec{A}$.
Notice that $\frac{\vec{A}}{\mathrm{k}}=\frac{1}{\mathrm{k}} \vec{A}$.
2) $\quad \mathbf{k} \vec{A} / / \vec{A} \quad$ i.e. the vector $\mathbf{k} \vec{A}$ is parallel to the vector $\vec{A}$ and with same direction if $\mathbf{k}>\mathbf{0}$ (positive) and with opposite direction if $\mathbf{k}<0$ (negative). (see the figure. (6.11))

3) If $\mathbf{k}=0$ then $\mathbf{k} \vec{A}=\overrightarrow{0}$ ( null vector ) i.e. $0 \vec{A}=\overrightarrow{0}$
4) The notation $-\vec{A}$ means that $\mathrm{k}=-1$
then $-\vec{A}=(-1) \vec{A}=-1(\vec{A})$.
that is $-1(\vec{A})=-\vec{A}$ then $|-1(\vec{A})|=|-\overrightarrow{\mathrm{A}}|$
$\therefore|-1(\overrightarrow{\mathrm{~A}})|=|-\overrightarrow{\mathrm{A}}|=|\vec{A}| \therefore|\vec{A}|=|-\vec{A}|$.
5) (1) $\vec{A}=1(\vec{A})=1 \vec{A}=\vec{A}$.
6) The vectors $\overrightarrow{\mathrm{A}}$ and $\vec{B}$ are said to be parallel if $\vec{A}=\mathrm{k} \vec{B}$ for some real number $\mathrm{k} \neq 0$. i.e. $\vec{A}=\mathrm{k} \vec{B} \Longrightarrow \vec{A} / / \vec{B} \Longrightarrow|\vec{A}|=|k \vec{B}|=|k||\vec{B}|$, but $|\vec{A}|=|k||\vec{B}|$ does not imply $\vec{A}=\mathrm{k} \vec{B}$ i.e. $|\vec{A}|=|k||\vec{B}| \nRightarrow \vec{A}=k \vec{B}$.
7) $\left.\quad\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right) \vec{A}=\mathrm{k}_{1} \vec{A}+\mathrm{k}_{2} \vec{A}\right\} \quad$ where $\mathrm{k}_{1} \mathrm{k}_{2}$ are $\left.\mathrm{k}_{1}\left(\mathrm{k}_{2} \vec{A}\right)=\left(\mathrm{k}_{1} \mathrm{k}_{2}\right) \vec{A} \quad\right\} \quad$ real numbers.

### 6.2.2. Addition of vectors:

If $\vec{A}=\left[x_{1}, y_{1}\right]$ and $\vec{B}=\left[x_{2}, y_{2}\right]$ are two plane vectors, the sum of these two vectors $\vec{A}+\vec{B}$ has been defined algebraically to be the vector:
$\vec{A}+\vec{B}=\left[x_{1}+x_{2}, y_{1}+y_{2}\right]$, in three - dimensions:
$\vec{A}+\vec{B}=\left[x_{1}+x_{2}, y_{1}+y_{2}, z_{1}+z_{2}\right]$ this operation can be interpreted geometrically as follows: let the vectors $\vec{A}$ and $\vec{B}$ be represented by line segment, so placed that the initial (tail ) point of B is placed at the terminal ( tip ) point of A , then join the initial point of A to the terminal point of $B$ to get a triangle, then the sum - vector $A+B$ is the vector whose tail is at the tail of A and whose tip is at the tip of B . (as shown in the figure. (6.12)), from the figure. (6.13) we can note that:

$$
\vec{A}+\vec{B}=\left[x_{1}, y_{1}\right]+\left[x_{2}, y_{2}\right]=\left[x_{1}+\boldsymbol{y}_{1}, \boldsymbol{x}_{2}+\boldsymbol{y}_{2}\right] \quad Y
$$



Fig.(6.12)


Fig.(6.13)

## Notes:

1) The sum of two vectors $\vec{A}$ and $\vec{B}$ can be obtained geometrically by the triangle rule as shown in the figure (6.12) in which the vectors $\vec{A}, \vec{B}$ and $\vec{A}+\vec{B}$ are represented by the three sides of a triangle, which shows that in general:
$|\vec{A}+\vec{B}|<|\vec{A}|+|\vec{B}|$ Which is called triangle in - equality.
2) $\quad|\vec{A}+\vec{B}|=|\vec{A}|+|\vec{B}|$ this relation holds only if $\vec{A}$ and $\vec{B}$ have the same direction and sense. (see the figure. (6.13)).


Fig.(6.12)
3) The sum of two vectors $\vec{A}$ and $\vec{B}$ can be obtained with an equivalent rule to triangle rule called the purallelogram rule Fig. (6.14).
In which the two vectors $\vec{A}$ and $\vec{B}$ are both drawn with the same initial (tial) point then the sum $\vec{A}+\vec{B}$ is represented by the diagonal of the parallelogram having adjacent sides representing $\vec{A}$ and $\vec{B}$.
This rule is obtained by completing the triangle to a parallelgram. (see the figure. (6.14)).


Fig.(6.14)
4) The parallelogram rule is used mainly in phisics and mechanics in combining the vector quantities such as forces, velocities, accelerators in which the sum $\vec{A}+\vec{B}$ is called the resultant.
5) Figure (6.14) shows also that $\vec{A}+\vec{B}=\vec{B}+\vec{A}$ which is known as commutative law of addition.
6) The algebraic form of the sum of two vectors in two - dimensions (inplane) can be discovered geometrically by using squared (graph) paper in a process steping from special cases to a general case, as shown in the figure.(6.15). by consedering the initial point of each vector as an origion .


Fig.(6.15)

## Example :

If $\vec{A}=[3,2]$ and $\vec{B}=[3,3]$ from the graph $\vec{A}+\vec{B}$ will be [6,5], thus through some examples such as the above we will deduce the general case that if
$\vec{A}=\left[x_{1}, y_{1}\right], \vec{B}=\left[x_{2}, y_{2}\right]$ then the sum:
$\vec{A}+\vec{B}=\left[x_{1}+x_{2}, y_{1}+y_{2}\right]$ which is shown symbolically in Fig. (6.13).
7) The basic properties of vector and geometrically addition are:

- The property $\vec{A}+\vec{B}=\vec{B}+\vec{A}$ is called the commutative property of addition of vectors.
- The property $\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$ is called the associative poperty of addition of vectors.
- $\quad \mathrm{k}(\vec{A}+\vec{B})=\mathrm{k} \vec{A}+\mathrm{k} \vec{B}$ is called the distributive property of vectors.
- $\overrightarrow{0}+\vec{A}=\vec{A}+\overrightarrow{0}=\vec{A}, \overrightarrow{0}$ is called the additive identity.

8) The sum of more than two vectors (n-vectors) may also be obtained geometrically by constructing a diagram ( polygon ) of line - segments representing the added vectors in which the end point (terminal point ) of each vector is the intia point of the next one, the sum then is represented by
the line segment joining the firist initial point to the last end point. (see the figure.(6.16).


Fig.(6.16)
If $\vec{A}=\left[x_{1}, y_{1}\right], \vec{B}=\left[x_{2}, y_{2}\right], \vec{C}=\left[x_{3}, y_{3}\right], \vec{D}=\left[x_{4}, y_{4}\right], \quad \vec{E}=\left[x_{5}\right.$ , $y_{5}$ ] then the sum algebraically is [ $x_{1}+x_{2}+x_{3}+x_{4}+x_{5}, y_{1}+y_{2}+y_{3}+$ $\left.y_{4}+y_{5}\right]$.
notice that the sum of vector is the resultant of these vectors and rotates the polygon in the opposite direction to that of the added vectors.
9) The unit vector $\vec{u}$ in the direction of the vector $\vec{A}$ is a vector having unit magnitude and the same direction and sense of $\vec{A}$. (see the figure.(6.17). $\vec{u}$ is obtained by the formula:

$$
\vec{u}=\frac{\vec{A}}{|\vec{A}|} ;|\vec{A}| \neq 0 \longrightarrow|\vec{u}|=1
$$



Fig.(6.17)

The rectangular unit vectors are those having the directions of the positive $\boldsymbol{x}$ , $\boldsymbol{y}$ and $\boldsymbol{z}$ axes and are denoted by $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ repectively, (see the figure.(6.18).
Thus:

$$
|\vec{\imath}|=|\vec{\jmath}|=|\vec{k}|=1 .
$$



In two -dimensions


In three - dimensions

Fig.(6.18)
Where $\vec{\imath}=[1,0,0], \vec{\jmath}=[0,1,0]$ and $\vec{k}=[0,0,1]$
If $\vec{A}=\left[x_{1}, y_{1}, z_{1}\right]$ is a three - dimensional vector then $x_{1}, y_{1}$ and $z_{1}$ are called $x, y$ and $z$ components of $\vec{A}$, (see the figure.(6.19).

and $x_{1} \vec{\imath}, y_{1} \vec{\jmath}$ and $z_{1} \vec{k}$ are called component vectors of $\vec{A}$ in the directions of $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$ axes respectively.
10) The vector $\vec{A}=\left[x_{1}, y_{1}, z_{1}\right]$ can be written as a sum of its component vectors as:
$\vec{A}=x_{1} \vec{\imath}+y_{1} \vec{\jmath}+z_{1} \vec{k}$
$|\vec{A}|=\sqrt{x_{1}^{2}+y_{1}^{2}+z_{1}^{2}}$
11) If $\vec{A}=\left[x_{1}, y_{1}, z_{1}\right], \vec{B}=\left[x_{2}, y_{2}, z_{2}\right]$ then
$\vec{A}=x_{1} \vec{\imath}+y_{1} \vec{\jmath}+z_{1} \vec{k} \quad, \vec{B}=x_{2} \vec{\imath}+y_{2} \vec{\jmath}+z_{2} \vec{k}$ the
$\vec{A}+\vec{B}=\left(x_{1}+x_{2}\right) \vec{\imath}+\left(y_{1}+y_{2}\right) \vec{\jmath}+\left(z_{1}+z_{2}\right) \vec{k}$ and
$|A+B|=\sqrt{\left(x_{1}+x_{2}\right)^{2}+\left(y_{1}+y_{2}\right)^{2}+\left(z_{1}+z_{2}\right)^{2}}$
$\mathrm{m} \vec{A}=\mathrm{m} x_{1} \vec{\imath}+\mathrm{m} y_{1} \vec{\jmath}+\mathrm{m} z_{1} \vec{k}$, where m is a real number.

### 6.2.3. Subtraction of vectors :

If $\vec{A}=\left[x_{1}, y_{1}\right], \vec{B}=\left[x_{2}, y_{2}\right]$ are two plane vectors, the difference of these two vectors $\vec{A}-\vec{B}$ has been defined algebraically to be the vector:
$\vec{A}-\vec{B}=\left[x_{1}-x_{2}, y_{1}-y_{2}\right]$. (see the figure.(6.20).
In three-dimensions $\vec{A}-\vec{B}=\left[x_{1}-x_{2}, y_{1}-y_{2}, z_{1}-z_{2}\right]$.
This opertion is interpreted geometracally by using the algebraic relalition:
$\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$ or $\vec{A}-\vec{B}=\vec{C} \quad \vec{A}=\vec{B}+\vec{C}$ or $\quad \vec{C}=\vec{A}+(-\vec{B})$


$$
=\overrightarrow{\mathrm{A}}+(-\overrightarrow{\mathrm{B}})
$$

Fig.(6.20)

## Notes:

1) $\vec{A}-\vec{A}=\overrightarrow{0}$
2) $\vec{A}-\overrightarrow{0}=\vec{A}$
3) $\overrightarrow{0}-\overrightarrow{\mathrm{A}}=-\vec{A}$
4) $\quad-\mathrm{k}(\vec{A})=\mathrm{k}(-\vec{A})=-\mathrm{k} \vec{A}, \mathrm{k} \in R$
5) $\vec{A}-(-\vec{B})=\vec{A}+\vec{B}$.
6) If $\vec{a}$ and $\vec{b}$ are the position vectors of the points A and B respectively Figure. (6.21) shows that :

$$
\vec{a}+\overrightarrow{A B}=\vec{b} \longrightarrow \overrightarrow{A B}=\vec{b}-\vec{a} .
$$

This relationship is very useful in dealing with many geometric relations and proofs.

i.e if $\mathrm{A}\left(x_{1}, y_{1}\right)$ and $\mathrm{B}\left(x_{2}, y_{2}\right)$ are two points in the plane then their position vectors are $\vec{a}=\left[x_{1}, y_{1}\right]$ and $\vec{b}=\left[x_{2}, y_{2}\right]$ respectively then the vector
$\overrightarrow{A B}=\vec{b}-\vec{a}=\left[x_{2}-x_{1}, y_{2}-y_{1}\right]$.
7) if $\vec{A}=\left[x_{1}, y_{1}, z_{1}\right]$ and $\mathrm{B}=\left[x_{2}, y_{2}, z_{2}\right]$ then

$$
\vec{A}=x_{1} \vec{\imath}+y_{1} \vec{\jmath}+z_{1} \vec{k}, \vec{B}=x_{2} \vec{\imath}+y_{2} \vec{\jmath}+z_{2} \vec{k}
$$

Then. $\vec{A}-\vec{B}=\left(x_{1}-x_{2}\right) \vec{\imath}+\left(y_{1}-y_{2}\right) \vec{\jmath}+\left(z_{1}-z_{2}\right) \vec{k}$

$$
|\vec{A}-\vec{B}|=\sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}+\left(z_{1}-z_{2}\right)^{2}}
$$

### 6.2.4 .The scalor product ( dot product

If $\overrightarrow{\boldsymbol{A}}$ and $\overrightarrow{\boldsymbol{B}}$ are two non-zero vectors and $\theta$ the angle between them (Figue. (6.22)). then the scalar product of these two vectors denoted by A.B is defined geometrically as $\overrightarrow{\boldsymbol{A}} \cdot \overrightarrow{\boldsymbol{B}}=|\overrightarrow{\boldsymbol{A}}||\overrightarrow{\boldsymbol{B}}| \cos \theta, 0 \leq \theta \leq \pi$


Fig.(6.22)

## Notes:

1) The result of $\vec{A} \cdot \vec{B}$ is a scalar as indicated by the name thus it is called scalar product.
2) $\quad \vec{A} \cdot \vec{B}$ is also called the dot product or inner product with respect to the $\operatorname{dot}($.$) used to denot this operation and read as (\mathrm{A} \operatorname{dot} \mathrm{B})$.
3) If $\vec{A}$ and $\vec{B}$ are non zero vectors then $|\vec{A}|$ and $|\vec{B}|$ are not zero, then

$$
\vec{A} \cdot \vec{B}=|\vec{A}||\vec{B}| \cos \theta=0 \Longrightarrow \cos \theta=0 \Longrightarrow \theta=90^{\circ}
$$

Which indicates that $\vec{A} \perp \vec{B}$ i.e $\vec{A}$ and $\vec{B}$ are perpendicular to each other.
4) The basic properties of dot product are :

- $\vec{A} \cdot \vec{B}=\vec{B} \cdot \vec{A}$, called the commutative property.
- $\quad \vec{A} \cdot(\vec{B} \cdot \vec{C})$ is a non defined opertion. i.e there is no such opertion.
- $\quad \vec{A} \cdot(\vec{B}+\vec{C})=\vec{A} \cdot \vec{B}+\vec{A} \cdot \vec{C}$ called the distributive property.
- $\quad m(\vec{A} \cdot \vec{B})=(m \vec{A}) \cdot \vec{B}=\vec{A} \cdot(m \vec{B})=(\vec{A} \cdot \vec{B}) m$. where $\boldsymbol{m}$ is a real number.

5) The following table shows the relations between the rectangular unit vectors with respect to the dot product, operation which shows that:
$\vec{\imath} \cdot \vec{\imath}=\vec{\jmath} \cdot \vec{\jmath}=\vec{k} \cdot \vec{k}=1$ and $\vec{\imath} \cdot \vec{\jmath}=\vec{\imath} \cdot \vec{k}=\vec{\jmath} \cdot \vec{k}=0, \vec{\imath}, \vec{\jmath}, \vec{k} \quad$ being perpendicular.

|  | $\vec{\imath}$ | $\vec{\jmath}$ | $\vec{k}$ |
| :--- | :--- | :--- | :--- |
| $\vec{\imath}$ | 1 | 0 | 0 |
| $\vec{\jmath}$ | 0 | 1 | 0 |
| $\vec{k}$ | 0 | 0 | 1 |

6) If $\vec{A}=\left[x_{1}, y_{1}, z_{1}\right]=x_{1} \vec{\imath}+y_{1} \vec{\jmath}+z_{1} \vec{k} \quad$ and $\vec{B}=\left[x_{2}, y_{2}, z_{2}\right]=x_{2} \vec{\imath}+y_{2} \vec{\jmath}+z_{2} \vec{k}$
Then by using the commutitivity and distributivity properties of dot product we can obtain an algebraic dot product formula which makes it easy to find the dot product of two vectors algebraically.

- $\quad \vec{A} \cdot \vec{B}=x_{1} x_{2}+y_{1} y_{2}+z_{1} z_{2}$.
- $\quad \vec{A} \cdot \vec{A}=|\vec{A}||\vec{A}|=|\vec{A}|^{2}=x_{1}^{2}+y_{1}^{2}+z_{1}^{2}$
$\vec{B} \cdot \vec{B}=|\vec{B}||\vec{B}|=|\vec{B}|^{2}=x_{2}^{2}+y_{2}^{2}+z_{2}^{2}$

7) The angle $\theta$ between two non zero vectors $A$ and $B$ is obtain by the formula:
$\cos \theta=\frac{\overrightarrow{\mathrm{A}} \cdot \overrightarrow{\mathrm{B}}}{|\overrightarrow{\mathrm{A}}||\overrightarrow{\mathrm{B}}|}$. which is obtained from the definition of the dot produt $\vec{A} \cdot \vec{B}=|\vec{A}||\vec{B}| \cos \theta$. $\cos \theta=\frac{\overrightarrow{\mathrm{A}} \cdot \overrightarrow{\mathrm{B}}}{|\overrightarrow{\mathrm{A}}||\overrightarrow{\mathrm{B}}|} \longrightarrow \theta=\cos ^{-1} \theta \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$ which indicates that if $\quad \theta=90^{\circ}$. i.e $\vec{A} \perp \vec{B}$ then $\vec{A} \cdot \vec{B}=0$ since $|\vec{A}||\vec{B}| \neq 0$, due to that $\vec{A}$ and $\vec{B}$ are non zero vectors.

### 6.2.5. Vector product ( cross product )

If $\vec{A}$ and $\vec{B}$ are two vectors and $\theta$ is the angle between then the vector product of these two vectors denoted by $\vec{A} \times \vec{B}$ is defined geometrically
$\vec{A} \times \vec{B}=|\vec{A}| \quad|\vec{B}| \sin \theta \quad, 0 \leq \theta \leq \pi$, where $\vec{u}$ is a unit vector normal (perpenticular) to the plane of A and B , as shown in the figure (6.23):


Fig.(6.23)

## Notes:

1) The result of $\vec{A} \times \vec{B}$ is a vector as indicated by the name, thus it is called vector product.
2) $\vec{A} \times \vec{B}$ is also called the cross product or outer product with repect to the symbol cross $(\times)$ used to denote this opertion, and read as (A cross B).
3) $\vec{u}$ is a unit vector normal to the plane of $\vec{A}$ and $\vec{B}$ i.e $\vec{u}$ is (perpendicular ) to both $\vec{A}$ and $\vec{B}$.
4) The direction of $\vec{u}$ is defined as the direction of advance of a right handed screw as vector $\vec{A}$ is turned into vector $\vec{B}$ and the angle of rotation is less than $\pi$.
5) The unit vector $\vec{u}$ indicates the direction of the vector $\vec{A} \times \vec{B}$.
6) The magnitude of the vector $\vec{A} \times \vec{B}$ is
$|\vec{A} \times \vec{B}|=|\vec{A}||\vec{B}| \sin \theta|\vec{u}| \ldots$. Since $|\vec{u}|=1$.
7) The direction of the vector $\vec{A} \times \vec{B}$ is perpendicular (normal) to both $\vec{A}$ and $\vec{B}$ i.e normal to the plane of $\vec{A}$ and $\vec{B}$.(see the figure (6.23)).
8) If $\vec{A}=\vec{B}$ or $\vec{A}$ is parallel to $\vec{B}(\vec{A} / / \vec{B})$ then $\theta=0$ and $\sin \theta=0$ then $\vec{A} \times \vec{B}=0$.
$\therefore$ if $\vec{A} \times \vec{B}=0$ and $\vec{A}$ and $\vec{B}$ are non zero vectors then $\vec{A}$ and $\vec{B}$ are parallel $(\vec{A} / / \vec{B})$.
9) If $\vec{A}=\left[x_{1}, y_{1}, z_{1}\right]=x_{1} \vec{\imath}+y_{1} \vec{\jmath}+z_{1} \vec{k}$

And $\vec{B}=\left[x_{2}, y_{2}, z_{2}\right]=x_{2} \vec{\imath}+y_{2} \vec{\jmath}+z_{2} \vec{k} \quad$ then $\vec{A} \times \vec{B}$ is defined algebraically as:

$$
\vec{A} \times \vec{B}=\left|\begin{array}{ccc}
\overrightarrow{1} & \vec{\jmath} & \vec{k} \\
x_{1} & y_{1} & z_{1} \\
x_{2} & y_{2} & z_{2}
\end{array}\right|
$$

The evaluation of this determinant gives:
$\vec{A} \times \vec{B}=\left(y_{1} z_{2}-y_{2} z_{1}\right) \vec{\imath}+\left(z_{1} x_{2}-z_{2} x_{1}\right) \vec{\jmath}+\left(\mathrm{x}_{1} y_{2}-x_{2} y_{1}\right) \vec{k}$
10) The following table shows the relations between the unit vectors $i, j$ , k with respect to the cross product operation.

Which shows that $\vec{\imath} \times \vec{\jmath}=\vec{\jmath} \times \vec{\jmath}=\vec{k} \times \vec{k}=0$.

| $\times$ | $\vec{\imath}$ | $\vec{\jmath}$ | $\vec{k}$ |
| :---: | :---: | :---: | :---: |
| $\vec{\imath}$ | $\overrightarrow{0}$ | $\vec{k}$ | $-\vec{\jmath}$ |
| $\vec{\jmath}$ | $-\vec{k}$ | $\overrightarrow{0}$ | $\vec{\imath}$ |
| $\vec{k}$ | $\vec{\jmath}$ | $-\vec{\imath}$ | $\overrightarrow{0}$ |

11) The basic properties of cross product are:

- $\vec{A} \times \vec{B}=-(\vec{B} \times \vec{A})$ i.e, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$.
i.e the cross product is not commutative.
* $\overrightarrow{\mathrm{A}} \times(\overrightarrow{\mathrm{B}}+\vec{C})=\vec{A} \times \vec{B}+\vec{A} \times \vec{C}$ distributive property.
* $(\vec{A}+\vec{B}) \times \vec{C}=\vec{A} \times \vec{C}+\vec{B} \times \vec{C}$ distributive property.
* $(\boldsymbol{M} \vec{A}) \times \vec{B}=\boldsymbol{M}(\vec{A} \times \vec{B})$, where $\boldsymbol{M}$ is areal number.
* $\vec{A} \times(\vec{B} \times \vec{C}) \neq(\vec{A} \times \vec{B}) \times \vec{C}$, i.e the associative property does not hold for all vectors under the cross product operation.


### 6.2.6. Triple products

Triple product deals with the multiplication of three vectors $\vec{A}, \vec{B}$ and $\vec{C}$, which discribes many combinations of scalar and vector multiplication, some of these are possible and meaningful and some are not.

The simplist particular combinations are :

## I ) The scalar triple product:

Let $\vec{A}, \vec{B}$ and $\vec{C}$ be any three vectors the combination $\vec{A} \cdot(\vec{B} \times \vec{C})$ denotes an operation called the scalar triple product of $\vec{A}, \vec{B}, \vec{C}$ in this order and is denoted by $[\vec{A}, \vec{B}, \vec{C}]=\vec{A} \cdot(\vec{B} \times \vec{C})$.

## Notes:

1) The result of the scalar triple product $\vec{A} \cdot(\vec{B} \times \vec{C})$ is a scalar as indicatal by the name, thus it is called scalar triple product.
2) The scalar triple product is frequently written as
$\vec{A} \cdot(\vec{B} \times \vec{C})=[\vec{A}, \vec{B}, \vec{C}]$ in the same order.
3) If $\vec{A}=\left[x_{1}, y_{1}, z_{1}\right]=x_{1} \vec{\imath}+y_{1} \vec{\jmath}+z_{1} \vec{k}$ and $\vec{B}=\left[x_{2}, y_{2}, z_{2}\right]=x_{2} \vec{\imath}+y_{2} \vec{\jmath}+z_{2} \vec{k} \quad$ and $\vec{C}=\left[x_{3}, y_{3}, z_{3}\right]=x_{3} \vec{\imath}+y_{3} \vec{\jmath}+z_{3} \vec{k} \quad$ then
$\vec{A} \cdot(\vec{B} \times \vec{C})$ can be calculated algebriacally as:
$[\vec{A}, \vec{B}, \vec{C}]=\vec{A} \cdot(\vec{B} \times \vec{C})=\left|\begin{array}{lll}x_{1} & y_{1} & z_{1} \\ x_{2} & y_{2} & z_{2} \\ x_{3} & y_{3} & z_{3}\end{array}\right|$

The evaluation of this determinant gives:

$$
\vec{A} \cdot(\vec{B} \times \vec{C})=x_{1}\left(y_{2} z_{3}-z_{2} y_{3}\right)+y_{1}\left(z_{2} x_{3}-x_{2} z_{3}\right)+z_{1}\left(x_{2} y_{3}-y_{2} z_{3}\right)
$$

4) The main properties of the scalar triple product indicate that any cyclic permutation (cyclic interchange) of the vectors $\vec{A}, \vec{B}, \vec{C}$ in the scalar triple product will give the same result, while any cyclic premutation (interchange) of two vectors gives the negative of the result, as follows:

- $\vec{A} \cdot(\vec{B} \times \vec{C})=(\vec{B} \times \vec{C}) \cdot \vec{A}$
- $\vec{A} \cdot(\vec{B} \times \vec{C})=\vec{B} \cdot(\vec{C} \times \vec{A})=\vec{C} \cdot(\vec{A} \times \vec{B})$

$$
=-\vec{A} \cdot(\vec{C} \times \vec{B})=-\vec{B} \cdot(\vec{A} \times \vec{C})=-\vec{C} \cdot(\vec{B} \times \vec{A})
$$

Which can be written as:
$[\vec{A}, \vec{B}, \vec{C}]=[\vec{B}, \vec{C}, \vec{A}]=[\vec{C}, \vec{A}, \vec{B}]=-[\vec{A}, \vec{C}, \vec{B}]=-[\vec{B}, \vec{A}, \vec{C}]$

$$
=-[\vec{C}, \vec{B}, \vec{A}] .
$$

5) $\vec{A} \cdot(\vec{B} \times \vec{C})=0$ implies that $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar i.e they lie inparallel planes.
II) The vector triple product:

Let $\vec{A}, \vec{B}$ and $\vec{C}$ be any three vectors, the combination $\vec{A} \times(\vec{B} \times \vec{C})$ denotes an operation called the vector triple product of $\vec{A}, \vec{B}, \vec{C}$.

## Notes:

1) The result of the vector triple product $\vec{A} \times(\vec{B} \times \vec{C})$ is a vector as indicated by the name thus it is called vector triple product.
2)     - The geometric interpretation of vector triple product $\vec{A} \times(\vec{B} \times \vec{C})$ Figure. (6.24) shows that since ( $\vec{B} \times \vec{C}$ ) is a vector normal to the plane of $\overrightarrow{\mathrm{B}}$ and $\vec{C}$, and $\overrightarrow{\mathrm{A}} \times(\overrightarrow{\mathrm{B}} \times \overrightarrow{\mathrm{C}})$ is normal to both vectors $\vec{A}$ and $(\overrightarrow{\mathrm{B}} \times \overrightarrow{\mathrm{C}})$ then $\vec{A} \times(\vec{B} \times \vec{C})$ must lie in the plane of $\vec{B}$ and $\vec{C}$.


Fig.(6.24)
3) - The main properties of vector triple productave are:

- $\vec{A} \times(\vec{B} \times \vec{C}) \neq(\vec{A} \times \vec{B}) \times \vec{C}$ this indicates that the associative property fails in the case of vector triple product showing the need for parentheses in $\vec{A} \times \vec{B} \times \vec{C}$ to a void a mbiguity.
- $\quad \vec{A} \times(\vec{B} \times \vec{C})=(\vec{A} . \vec{C}) \vec{B}-(\vec{A} . \vec{B}) \vec{C} \quad$ and
$\vec{B} \times(\vec{C} \times \vec{A})=(\vec{B} \cdot \vec{A}) \vec{C}-(\vec{B} \cdot \vec{C}) \vec{A}$
$(\vec{A} \times \vec{B}) \times \vec{C}=(\vec{A} \cdot \vec{C}) \vec{B}-(\vec{B} \cdot \vec{C}) \vec{A} \quad$ and
$\vec{C} \times(\vec{A} \times \vec{B})=(\vec{C} \cdot \vec{B}) \vec{A}-(\vec{C} \cdot \vec{A}) \mathrm{B}$
- $\quad(\vec{A} \times \vec{B}) \times \vec{C}=\vec{C} \times(\vec{B} \times \vec{A})$

4)     - Products involve more than three vectors may be expanded using the scalar and vector triple products as follows :

- $\quad(\vec{A} \times \vec{B}) .(\vec{C} \times \vec{D})=(\vec{A} . \vec{C})(\vec{B} . \vec{D})-(\vec{A} . \vec{D})(\vec{B} . \vec{C})$ which indicates that the result is a scalar.
- $(\vec{A} \times \vec{B}) \times(\vec{C} \times \vec{D})=[\vec{A}, \vec{B}, \vec{D}] \vec{C}-[\vec{A}, \vec{B}, \vec{C}] \vec{D}$

$$
\begin{aligned}
& =\mathrm{A} \cdot(\mathrm{~B} \times \mathrm{D}) \mathrm{C}-\mathrm{A} \cdot(\mathrm{~B} \times \mathrm{C}) \mathrm{D} \\
& =-[\vec{C}, \vec{D}, \vec{B}] \vec{A}+[\vec{C}, \vec{D}, \vec{A}] \vec{B} \\
& =-\vec{C} \cdot(\vec{D} \times \vec{B}) \vec{A}+\vec{C} \cdot(\vec{D} \times \vec{A}) \vec{B}
\end{aligned}
$$

Notice that the result is a vector .
5) - There are many possibilities for products involving three vectors some of them are possible and meaningful and some are not that is some of these are defined and some are undefined such as:

- $\quad \vec{A}(\vec{B} \cdot \vec{C})=(\vec{B} \cdot \vec{C}) \vec{A}$ the result is a vector.
- $\vec{A} \cdot(\vec{B} \times \vec{C})=[\vec{A}, \vec{B}, \vec{C}]$ the result is a scalar
- $\vec{A} \times(\vec{B} \times \vec{C})$ the result is a vector.
- $\quad \vec{A} \cdot(\vec{B} \cdot \vec{C})$ is un defined.
- $\quad \vec{A} \times(\vec{B} . \vec{C})$ is un defined.


## Examples:

1)     - Put each of the following vectors in the $\vec{\imath}, \vec{\jmath}, \vec{k}$ components form:
$\vec{A}=[-2,3,-4], \vec{B}=[3,0,-2]$
$\vec{C}=[1,-3,0], \vec{D}=[0,2,5]$.
Solution : $\vec{A}=[-2,3,-4]=-2 \vec{\imath}+3 \vec{\jmath}-4 \vec{k}$

$$
\vec{B}=[3,0,-2]=3 \vec{\imath}-2 \vec{k}
$$

$$
\mathrm{C}=[1,-3,0]=\vec{\imath}-3 \vec{\jmath}, \mathrm{D}=[0,2,5]=2 \vec{\jmath}+5 \vec{k}
$$

2)     - Put each of the following vectors in the cartesian components form:
$\mathrm{A}=3 \vec{\imath}-2 \vec{\jmath}+\vec{k} \quad, \mathrm{~B}=-2 \vec{\imath}+3 \vec{\jmath}$
$\mathrm{C}=\vec{\imath}-3 \vec{k}$
$\mathrm{D}=\vec{\jmath}+2 \vec{k}$

## Solution:

$\mathrm{A}=3 \vec{\imath}-2 \vec{\jmath}+\vec{k}=[3,-2,1]$
$\vec{B}=-2 \vec{\imath}+3 \vec{\jmath}=[-2,3,0], \mathrm{C}=\mathrm{i}-3 \mathrm{k}=[1,0,-3]$
$\mathrm{D}=\vec{\jmath}+2 \vec{k}=[0,1,2]$.
3) - If $\vec{A}=2 \vec{\jmath}-3 \vec{\imath}+\vec{k} \quad, \mathrm{~B}=-\vec{\imath}+4 \vec{k}-3 \vec{\jmath}$
$\mathrm{C}=3 \vec{k}-2 \vec{\jmath}, \mathrm{D}=-3 \vec{\jmath}-\vec{\imath}$.
Perform each of the following operations $\vec{A}+\vec{B}, 2 \vec{C}+3 \vec{D}, \vec{B}-\vec{A},|\vec{A}|$, $|\vec{D}|$
Solution : notice that in case of $\vec{\imath}, \vec{\jmath}, \vec{k}$ component form the order of components may be written in any form due to the commutative property of the sum of vectors, thus it might be more convenient to put the vectors in the cartesian form as follows:

$$
\begin{aligned}
& \vec{A}=2 \vec{\jmath}-3 \vec{\imath}+\vec{k}=[-3,2,1] \\
& \vec{B}=-\vec{\imath}+4 \vec{k}-3 \vec{\jmath}=[-1,-3,4] \\
& \vec{C}=3 \vec{k}-2 \vec{\jmath}=[0,-2,3] \\
& \vec{D}=-3 \vec{\jmath}-\vec{\imath}=[-1,-3,0] \\
& \vec{A}+\vec{B}=[-3,2,1]+[-1,-3,4]=[-4,-1,5] \\
& \vec{B}-\vec{A}=[-1,-3,4]-[-3,2,1]=[2,-5,3] \\
& 2 \vec{C}+3 \vec{D}=2[0,-2,3]+3[-1,-3,0] \\
& \quad=[0,-4,6]+[-3,-9,0] \\
& \quad=[-3,-13,6]
\end{aligned} \begin{aligned}
&|\vec{A}|=\sqrt{(-3)^{2}+(2)^{2}+(1)^{2}} \\
& \quad=\sqrt{9+4+1}=\sqrt{14} \\
&|\vec{D}|= \sqrt{(-1)^{2}+(-3)^{2}+(0)^{2}} \\
& \quad=\sqrt{1+9+0}=\sqrt{10}
\end{aligned}
$$

4)     - Find the angle between the two vectors
$\vec{A}=2 \vec{\imath}+2 \vec{\jmath}-\vec{k} \quad$ and $\quad \vec{B}=6 \vec{\imath}-3 \vec{\jmath}+2 \vec{k}$

## Solution:

$\vec{A} \cdot \vec{B}=|\vec{A}||\vec{B}| \cos \theta, \quad \cos \theta=\frac{\overrightarrow{\mathrm{A}} \cdot \overrightarrow{\mathrm{B}}}{|\overrightarrow{\mathrm{A}}||\overrightarrow{\mathrm{B}}|}$
$|\vec{A}|=\sqrt{(2)^{2}+(2)^{2}+(-1)^{2}}=3$
$\vec{B}=\sqrt{(6)^{2}+(-3)^{2}+(2)^{2}}=7$
$\vec{A} \cdot \vec{B}=2(6)+2(-3)+(-1)(2)$
$=12-6-2$
$=12-8$

$$
=4
$$

Then $\cos \theta=\frac{4}{(3)(7)}=\frac{4}{21}=0.1905$
$\therefore \theta=\cos ^{-1} \frac{4}{21}$ or $\theta=79^{\circ}$ approximately
5) -Show that the vector $\mathrm{A}=3 \vec{\imath}-2 \vec{\jmath}+\vec{k} \quad, \mathrm{~B}=2 \vec{\imath}+\vec{\jmath}-4 \vec{k} \quad$ are perpendicular
Solution:
Put the vector $\vec{A}$ and $\vec{B}$ in the cartesian components form
$\vec{A}=[3,-2,1], \vec{B}=[2,1,-4]$

$$
\begin{aligned}
\therefore \vec{A} \cdot \vec{B} & =(3)(2)+(-2)(1)+(1)(-4) \\
& =6-2-4 \\
& =6-6 \\
& =0
\end{aligned}
$$

$\therefore \vec{A} \cdot \vec{B}=0 \longrightarrow \vec{A} \perp \vec{B}$.
6) - If $\vec{A}=2 \vec{\imath}-3 \vec{\jmath}-\vec{k} \quad, \quad \mathrm{~B}=\vec{\imath}+4 \vec{\jmath}-2 \vec{k}$

Find $\vec{A} \times \vec{B}, \vec{B} \times \vec{A}$
Solution:

$$
\begin{aligned}
(\vec{A} \times \vec{B}) & =\left|\begin{array}{ccc}
\vec{\imath} & \vec{\jmath} & \vec{k} \\
2 & -3 & -1 \\
1 & 4 & -2
\end{array}\right|=\vec{\imath}\left|\begin{array}{cc}
-3 & -1 \\
4 & -2
\end{array}\right|-\vec{\jmath}\left|\begin{array}{cc}
2 & -1 \\
1 & -2
\end{array}\right|+\vec{k}\left|\begin{array}{cc}
2 & -3 \\
1 & 4
\end{array}\right| \\
& =10 \vec{\imath}+3 \vec{\jmath}+11 \vec{k}=[10,3,11]
\end{aligned}
$$

$$
\begin{aligned}
(\vec{B} \times \vec{A})= & \left|\begin{array}{ccc}
\vec{\imath} & \vec{\jmath} & \vec{k} \\
1 & 4 & -2 \\
2 & -3 & -1
\end{array}\right|=\vec{\imath}\left|\begin{array}{cc}
4 & -2 \\
-3 & -1
\end{array}\right|-\vec{\jmath}\left|\begin{array}{cc}
1 & -2 \\
2 & -1
\end{array}\right|+\vec{k}\left|\begin{array}{cc}
1 & 4 \\
2 & -3
\end{array}\right| \\
& =-10 \vec{\imath}-3 \vec{\jmath}-11 \vec{k}=[-10,-3,-11]
\end{aligned}
$$

This indicates that $\vec{A} \times \vec{B}=-\vec{B} \times \vec{A}$
7) -If $\vec{A}=2 \vec{\imath}-3 \vec{\jmath}, \vec{B}=\vec{\imath}+\vec{\jmath}-\vec{k}$ and $\vec{C}=2 \vec{\imath}-\vec{k}$

Evaluate A. $(\mathrm{B} \times \mathrm{C})$.

## Solution:

$$
\begin{aligned}
\vec{A} \cdot(\vec{B} \times \vec{C})= & \left|\begin{array}{ccc}
2 & -3 & 0 \\
1 & 1 & -1 \\
2 & 0 & -1
\end{array}\right|=2\left|\begin{array}{cc}
1 & -1 \\
0 & -1
\end{array}\right|+3\left|\begin{array}{ll}
1 & -1 \\
2 & -1
\end{array}\right|+\left|\begin{array}{ll}
1 & 1 \\
2 & 0
\end{array}\right| \\
& =-2+3+0 \\
& =1
\end{aligned}
$$

8)     - If $\vec{A}=3 \vec{\imath}-\vec{\jmath}+2 \vec{k}, \vec{B}=2 \vec{\imath}+\vec{\jmath}-\vec{k}$

And $\vec{C}=\vec{\imath}-2 \vec{\jmath}+2 \vec{k}$ find $\vec{A} \times(\vec{B} \times \vec{C})$
Solution
$\overrightarrow{\mathrm{B}} \times \overrightarrow{\mathrm{C}}=\left|\begin{array}{ccc}\vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & -1 \\ 1 & -2 & 2\end{array}\right|=-5 \vec{\jmath}-5 \vec{k}$

$$
\begin{gathered}
\overrightarrow{\mathrm{A}} \times(\overrightarrow{\mathrm{B}} \times \overrightarrow{\mathrm{C}})=\left|\begin{array}{ccc}
\vec{\imath} & \vec{\jmath} & \vec{k} \\
3 & -1 & 2 \\
0 & -5 & -5
\end{array}\right|=15 \vec{\imath}+15 \vec{\jmath}-15 \vec{k} \\
=15[1,1,-1]
\end{gathered}
$$

If, $a(3,1,-1), b(5,4,-2), c(-1,2,4), d(-2,3,5)$ are four points show that $\overrightarrow{A B}$ and $\overrightarrow{C D}$ are perpendicular

## Solution:

$$
\begin{aligned}
& \therefore \overrightarrow{A B}=\vec{b}-\vec{a}=[5,4,-2]-[3,1,-1] \\
&=[2,3,-1] \\
& \overrightarrow{C D}=\vec{d}-\vec{c}=[-2,3,5]-[-1,2,4] \\
&=[-1,1,1] \\
& \therefore \overrightarrow{A B} \cdot \overrightarrow{C D}= {[2,3,-1] \cdot[-1,1,1]=-2+3-1 } \\
&=0
\end{aligned}
$$

$\therefore \overrightarrow{A B} \cdot \overrightarrow{C D}=0 \longrightarrow \overrightarrow{A B} \perp \overrightarrow{C D}$.

### 6.3.Voctor geometry

Vector geometry is the brarch of vector mathematics concorned with the vector approach which makes vectors the fundamental concept to study the euclidian plane, it is a geometry in which many abstrect theorems are proved by using vector methods. in this approach vectors also furnish a ameans of studying plane and solid anaytic geometriacal concepts and relationships, they introduce a sufficient degree of simplification in to the study. in this section we will introduce some geometrical problems tackled using vector methods.

### 6.3.1. Problems in geometry and trigonometry

In this section we will introduce some gemetrical problems tackled using vector methods.

1) -Three points $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear (lie on a straight line) if there is exits a real number $\boldsymbol{k} \neq 0$. See the figure (6.25)
the $\overrightarrow{A C}=\boldsymbol{k} \overrightarrow{A B}$


## Example

Show that the points $\overrightarrow{\boldsymbol{a}}(2,3,4), \overrightarrow{\boldsymbol{b}}(5,6,8)$ And $\overrightarrow{\boldsymbol{c}}(8,9,12)$ are collinear.
Solution : we first calculate $\overrightarrow{A B}$ and $\overrightarrow{A C}$.

$$
\begin{aligned}
\overrightarrow{A B} & =\vec{b}-\vec{a} & & \overrightarrow{A C}=\vec{c}-\vec{a} \\
& =[5,6,8]-[2,3,4] & & =[8,9,12]-[2,3,4] \\
& =[3,3,4] & & =[6,6,8]=2[3,3,4]
\end{aligned}
$$

Since $\overrightarrow{A C}=2 \overrightarrow{A B}$, we deduce that $\overrightarrow{\boldsymbol{a}}, \overrightarrow{\boldsymbol{b}}$, and $\overrightarrow{\boldsymbol{c}}$ are collinar.
2) - Division formula

This formula is concerned with finding the co-ordinates of a point dividing the distance between two points $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$ in a given ratio $\mathbf{L}: \mathbf{M}$. This formula
is also called section formula, or ratio formula, notice that the division can be internally or externally.
We will use the collinearity property, since $\overrightarrow{\boldsymbol{a}}, \overrightarrow{\boldsymbol{b}}$, and $\overrightarrow{\boldsymbol{c}}$ are collinear then $\overrightarrow{A C}=\mathrm{k} \overrightarrow{\mathrm{AB}}$ we notice that $\mathrm{k}=\frac{\mathrm{L}}{\mathrm{L}+\mathrm{M}}$ (see the figure (6.26)). Then:

$$
\begin{aligned}
\overrightarrow{A C} & =\frac{\mathrm{L}}{\mathrm{~L}+\mathrm{M}} \overrightarrow{A B} \longrightarrow \vec{c}-\vec{a}=\frac{\mathrm{L}}{\mathrm{~L}+\mathrm{M}}(\vec{b}-\vec{a}) \\
\therefore \vec{c} & =\frac{\mathrm{L}}{\mathrm{~L}+\mathrm{M}}(\vec{b}-\vec{a})+\vec{a} .
\end{aligned}
$$



Fig.(6.26)

$$
\begin{aligned}
& \vec{c}=\frac{\mathrm{L}}{\mathrm{~L}+\mathrm{M}} \vec{b}-\frac{\mathrm{L}}{\mathrm{~L}+\mathrm{M}} \vec{a}+\vec{a}=\frac{\mathrm{L} \overrightarrow{\mathrm{~b}}-\mathrm{L} \overrightarrow{\mathrm{a}}+(\mathrm{L}+\mathrm{M}) \overrightarrow{\mathrm{a}}}{\mathrm{~L}+\mathrm{M}}= \\
& \frac{\mathrm{L} \overrightarrow{\mathrm{~b}}-\mathrm{L} \vec{a}+\mathrm{L} \vec{a}+\mathrm{M} \vec{a}}{\mathrm{~L}+\mathrm{M}} \\
& \quad \therefore \vec{C}=\frac{\mathrm{L} \vec{b}+\mathrm{M} \vec{a}}{\mathrm{~L}+\mathrm{M}}=\frac{\mathrm{M} \vec{a}+\mathrm{L} \vec{b}}{\mathrm{M}+\mathrm{L}}
\end{aligned}
$$

The co-ordinates of the division point C are the components of its position vector $\vec{C}$. Notice that this formula is more general than the analytic geometry formula, it can be used in n - dimensional situation, and combines all co-ordinates ( $x$ and $y$ and $z$ ) in one formula, this indicates that the vector methods are independent of dimensions.

Example : find the co-ordinates of the point dividing the line - segment AB in the ratio $2: 3$ where $\mathrm{A}(2,3), \mathrm{B}(7,8)$. See the figure (6.27).

## Solution

Use the formula $\vec{C}=\frac{\mathrm{M} \vec{a}+\mathrm{L} \vec{b}}{\mathrm{M}+\mathrm{L}}$


Fig.(6.27)
$\therefore \vec{C}=\frac{3 \overrightarrow{\mathrm{a}}+2 \overrightarrow{\mathrm{~b}}}{3+2}$
$=\frac{3[2,3]+2[7,8]}{5}$
$=\frac{[20,25]}{5} \quad \therefore \vec{C}=[4,5] \longrightarrow \vec{C}(4,5)$
$\therefore$ The division point is $\vec{C}(4,5)$
3) Prove the theorem which sayes that:
" The line segemt joining the midpoints of two sides of a triangle is paallel to the third side and equals half of of its length. (Figure. (6.28)).


Fig.(6.28)

$$
\begin{aligned}
\overrightarrow{D E} & =\overrightarrow{D B}+\overrightarrow{B E} \\
& =\frac{1}{2} \overrightarrow{A B}+\frac{1}{2} \overrightarrow{B C} \\
& =\frac{1}{2}(\overrightarrow{A B}+\overrightarrow{B C})
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{1}{2} \overrightarrow{A C} \\
\therefore \overrightarrow{D E} & =\frac{1}{2} \overrightarrow{A C} \longrightarrow \overrightarrow{D E} / / \overrightarrow{A C} \text { and }|\overrightarrow{D E}|=\frac{1}{2}|\overrightarrow{A C}|
\end{aligned}
$$

- The provious theorem can be proved by another manner using the division formula as follows refering to the figure (6.28) we notice that two points D and E are the mid points of the two sides AB and BC i.e in the ratio $1: 1$ then if $d$ and $e$ are the position vectors of the points $D$ and $E$ respectively.
$\therefore \vec{d}=\frac{\vec{a}+\vec{b}}{2}, \vec{e}=\frac{\vec{b}+\vec{c}}{2}$
$\therefore \overrightarrow{D E}=\vec{e}-\vec{d}$

$$
\begin{aligned}
& =\frac{\vec{b}+\vec{c}}{2}-\frac{\vec{a}+\vec{b}}{2} \\
& \quad=\frac{\vec{b}+\vec{c}-\vec{a}-\vec{b}}{2}=\frac{\vec{c}-\vec{a}}{2}
\end{aligned}
$$

$\therefore \overrightarrow{A C}=\vec{c}-\vec{a}$
$\therefore \overrightarrow{D E}=\frac{1}{2} \overrightarrow{A C}$
$\therefore \overrightarrow{D E} / / \overrightarrow{A C}$ and $|\overrightarrow{D E}|=\frac{1}{2}|\overrightarrow{A C}|$.

- The previous thoores can be proved in a general case when the division ratio is $\mathrm{L}: \mathrm{M}$. As shown in the figure. (6.29) .


Fig.(6.29)

$$
\begin{aligned}
& \vec{d}= \frac{M \vec{a}+L \vec{b}}{L+M}, \vec{e}=\frac{M \vec{c}+L \vec{b}}{L+M} \\
& \overrightarrow{D E}=\vec{e}-\vec{d} \longrightarrow \overrightarrow{D E}=\frac{M \vec{c}+L \vec{b}}{L+M}-\frac{M \vec{a}+L \vec{b}}{L+M} \\
& \therefore \overrightarrow{D E}=\frac{M \vec{c}+L \vec{b}-M \vec{a}-L \vec{b}}{L+M} \\
&=\frac{M \vec{c}-M \vec{a}}{L+M}=\frac{M(\vec{c}-\vec{a})}{L+M} \\
& \therefore \overrightarrow{D E}=\frac{M}{L+M}(\vec{c}-\vec{a}) \longrightarrow \overrightarrow{D E}=\frac{M}{L+M} \overrightarrow{A C} \\
& \therefore \overrightarrow{D E}=\frac{M}{L+M} \overrightarrow{A C} \longrightarrow \overrightarrow{D E} / / \overrightarrow{A C} \text { and }|\overrightarrow{D E}|=\frac{M}{L+M}|\overrightarrow{A C}| .
\end{aligned}
$$

4) Prove the therem saying that:
" the diagonals of a parallogram bisect each other"
Proof : Associate vectors $\overrightarrow{V 1}$ and $\overrightarrow{V 2}$ with two adjacent sides of the parallelogram ABCD. See the figure. $(6.30-\mathrm{a}, \mathrm{b})$ :


Fig.(6.30-a,b)
i) If $\mathbf{P}$ is the mid point of the diagonal $\mathbf{A C}$ then :

$$
\overrightarrow{A P}=\frac{1}{2} \overrightarrow{A C}=\frac{1}{2}(\overrightarrow{V 1}+\overrightarrow{V 2})
$$

ii) If Q is the mid point of the diagonal BD then:

$$
\begin{aligned}
& \overrightarrow{A Q}=\overrightarrow{A D}+\overrightarrow{D Q}=\overrightarrow{A D}+\frac{1}{2} \overrightarrow{D B}=\overrightarrow{V 2}+\frac{1}{2}(\overrightarrow{V 1}-\overrightarrow{V 2}) \\
& =\overrightarrow{V 2}+\frac{1}{2} \overrightarrow{V 1}-\frac{1}{2} \overrightarrow{V 2} \longrightarrow \overrightarrow{A Q}=\frac{1}{2}(\overrightarrow{V 1}+\overrightarrow{V 2})
\end{aligned}
$$

$\overrightarrow{A P}=\overrightarrow{A Q} \longrightarrow \mathbf{P}$ and $\mathbf{Q}$ coincide thus the diagonals bisect each other.
5) Prove the theorem saying that:
" The medians of a triangle trisect each other"
In the figure.(6.31), let $\mathbf{D}, \mathbf{E}, \mathbf{F}$ be the mid points of the sides of the triangle ABC.


Fig.(6.31)

Then the line segments $\mathbf{A D}, \mathbf{B E}$ and $\mathbf{C F}$ are called the medians of the triangle ABC.
The theorem states that the medians intersect in a point ( $\mathbf{P}$ ) which trisect each median that is divides each median in the ratio $\mathbf{2 : 1}$ ( 2 from vectex and 1 from the base )
Proof : In the figure (6.32), let $\mathbf{P}$ be the point of trisection (called also the centroid of the triangle ) of the median $\mathbf{A D}$ nearest to $\mathbf{D}$.


Fig.(6.32)

And let the position vectors of the point $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and P be $\vec{a}, \vec{b}, \vec{c}$, $\vec{d}$ and $\vec{P}$ respectivel. using the general division formula we get: $\vec{P}=\frac{\vec{a}+2 \vec{d}}{3}$

Since $\mathbf{D}$ is mid - point of $\mathbf{B C}$ so ,

$$
\begin{gathered}
\vec{d}=\frac{\vec{b}+\vec{c}}{2} \\
\therefore \mathrm{P}=\frac{\vec{a}+2(\vec{b}+\vec{c}) / 2}{3}=\frac{\vec{a}+\vec{b}+\vec{c}}{3}
\end{gathered}
$$

In a similar way we can show that the position vectors of the points of trisection of the other two medians are also equal $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$, then from the symmetry of this result we deduce that the corresponding points of trisection of the three medians are coincide and hence that the medians of the triangle trisect each other then $\vec{P}=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$ the co-ordinates of the point $\mathbf{P}$ of trisection are the components of the vector $\vec{P}$ then the coordinates of the point of trisection are easily found for if the verticies of the triaugle ABC are $\mathbf{A}\left(x_{1}, y_{1}, z_{1}\right), \mathbf{B}\left(x_{2}, y_{2}, z_{2}\right), \mathbf{C}\left(x_{3}, y_{3}, z_{3}\right)$ then $\vec{a}=\left[x_{1}, y_{1}, z_{1}\right], \vec{b}=\left[x_{2}, y_{2}, z_{2}\right], \vec{c}=\left[x_{3}, y_{3}, z_{3}\right]$ then $\mathrm{P}=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$ written in full becomes $\vec{P}=\left[\frac{x_{1}+x_{2}+x_{3}}{3}, \frac{y_{1}+y_{2}+y_{3}}{3}, \frac{z_{1}+z_{2}+z_{3}}{3}\right]$
$\therefore \mathbf{P}\left(\frac{x_{1}+x_{2}+x_{3}}{3}, \frac{y_{1}+y_{2}+y_{3}}{3}, \frac{z_{1}+z_{2}+z_{3}}{3}\right)$ Is the point of trisection.
6) Prove the theorem which states that:
" If the mid points of the consecutive sides of any quodrilateral are connected by line segments the resulting quadrilateral is parrallelogram". Proof : let A, B , C, D. Fig.(6.33) be the verices of the given quadrilateral with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively. and $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ the mid points of the sides with position vectors $\vec{p}, \vec{q}, \vec{r}, \vec{s}$.

Refering to the figure.(6.33) and using the mid point formula we obtain:


Fig.(6.33)
$\vec{P}=\frac{1}{2}(\vec{a}+\vec{b}), \vec{q}=\frac{1}{2}(\vec{b}+\vec{c})$
$\vec{r}=\frac{1}{2}(\vec{c}+\vec{d}), \vec{s}=\frac{1}{2}(\vec{a}+\vec{d})$
$\therefore \overrightarrow{P Q}=\vec{q}-\vec{p}=\frac{1}{2}(\vec{b}+\vec{c})-\frac{1}{2}(\vec{a}+\vec{b})=\frac{1}{2}(\vec{c}-\vec{a})$
$\therefore \overrightarrow{S R}=\vec{r}-\vec{s}=\frac{1}{2}(\vec{c}+\vec{b})-\frac{1}{2}(\vec{a}+\vec{b})=\frac{1}{2}(\vec{c}-\vec{a})$
$\therefore \overrightarrow{P Q}=\overrightarrow{S R} \longrightarrow \overrightarrow{P Q} / / \overrightarrow{S R}$ and $|\overrightarrow{P Q}|=|\overrightarrow{S R}|$
$\therefore \mathrm{PQ} \mathrm{RS}$ is a paralleogram.
7) Prove the theorem which states that:
" the median to the base of an isosceles triangle is perpendicular to the base"

Proof: Associate vectors $\overrightarrow{V 1}, \overrightarrow{V 2}$ with the two equal sides of the isosceles triangle: A B C, then $|\overrightarrow{V 1}|=|\overrightarrow{V 2}|$.(see the figure (6.34)).


Fig.(6.34)
$\therefore \overrightarrow{C E}=\overrightarrow{V 3}=\frac{1}{2}(\overrightarrow{V 1}+\overrightarrow{V 2})$ (being half of the diagonal vector of the parallelogram if completed $)$.

The base vector $\overrightarrow{A B}=\overrightarrow{V 1}-\overrightarrow{V 2}$.

$$
\begin{aligned}
\therefore \overrightarrow{C E} \cdot \overrightarrow{A B} & =\overrightarrow{V 3} \cdot(\overrightarrow{V 1}-\overrightarrow{V 2}) \\
& =\frac{1}{2}(\overrightarrow{V 1}+\overrightarrow{V 2}) \cdot(\overrightarrow{V 1}-\overrightarrow{V 2}) \\
& =\frac{1}{2}(\overrightarrow{V 1} \cdot \overrightarrow{V 1}-\overrightarrow{V 1} \cdot \overrightarrow{V 2}+\overrightarrow{V 1} \cdot \overrightarrow{V 2}-\overrightarrow{V 2} \cdot \overrightarrow{V 2}) \\
& =\frac{1}{2}(\overrightarrow{V 1} \cdot \overrightarrow{V 1}-\overrightarrow{V 2} \cdot \overrightarrow{V 2}) \\
& =\frac{1}{2}\left(|\overrightarrow{V 1}|^{2}-|\overrightarrow{V 2}|^{2}\right)=0
\end{aligned}
$$

Thus $\overrightarrow{C E} \perp \overrightarrow{A B}$ i.e the median is perpendicular to the base in the isosceles triangle.

Notice that if the Figure (6.34) is completed we get a rhombus of which the diagonals are perpendicular thus $\overrightarrow{C D} \perp \overrightarrow{A B}$.(see the figure (6.35)).


Fig.(6.35)
8) Show that the area of a parallelogram with sides $V 1$ and $V 2\left|\mathrm{~V}_{1} \times \mathrm{V}_{2}\right|$. Associate vectors $\overrightarrow{V 1}$ and $\overrightarrow{V 2}$ with two adjacent sides of the given paralleogram with height (h). .(see the figure (6.36)).
Area of the parallelogram $=$ height $\times$ base
$\therefore$ Area $=\mathbf{h}|\overrightarrow{V 1}| \quad \therefore \mathbf{h}=|\overrightarrow{V 2}| \sin \theta$


Fig.(6.36)
$\therefore$ Area $=|V 1||V 2| \sin \theta$
$\therefore$ Area of parallelogram $=|\overrightarrow{V 1} \times \overrightarrow{V 2}|$.

- Notice that this formula applies to the rectangle and square and rhombus being special cases of the parallelogram.
- The area of the triangle with two sides $\overrightarrow{V 1}$ and $\overrightarrow{V 2}$ is $\frac{1}{2}|V 1 \times V 2|$.

9) Equation of a straight line it is indicated in the previous unit. 5 that the straight line in two - dimsions (in the plane ) may be complety specified by means of one point on the line and its slope. in vector analytic geometry the line in two and three dimensions may be completely specified by means of a non zero vector parallel to the line called direction vector and a point on the line.
10) in case of the line in two dimensions as shown in the figure (6.37), let $\vec{V}=[\mathrm{a}, \mathrm{b}]$ be the direction vector which is parrallel to $\mathbf{L}$ and $\mathbf{P}\left(x_{1}, y_{1}\right)$ be a point on $\mathbf{L}$ and $\mathbf{Q}(x, y)$ be an arbitrary (general ) point on $\mathbf{L}$. Refering to Fig.(6.40) we notice that $\overrightarrow{P Q} / / \vec{V} \longrightarrow \mathrm{PQ}=\mathrm{tv}$ where $(\mathbf{t})$ is a real number called parameter.
$\therefore \vec{q}-\vec{p}=t \vec{v} \longrightarrow\left[x-x_{1}, y-y_{1}\right]=\mathrm{t}[\mathrm{a}, \mathrm{b}]$
This equalion is called vector equation of $\mathbf{L}$.

$$
\left.\begin{array}{rl}
\therefore x-x_{1} & =\mathrm{at} \\
y-y_{1} & =\mathrm{bt} \quad x=x_{1}+\mathrm{at} \\
y=y_{1}+\mathrm{bt}
\end{array}\right\} \quad \begin{aligned}
& \text { called parametric } \\
& \text { equations of } \mathrm{L}
\end{aligned}
$$



Fig.(6.37)
If the components of the direction vector $\vec{V}$ are non zero. i.e.. $\vec{V}=[\mathrm{a}, \mathrm{b}] \neq 0$, the system of parametric equations can be written in the form

$$
\begin{aligned}
& x-x_{1}=\mathrm{at} \longrightarrow \mathrm{t}=\frac{x-x_{1}}{\mathrm{a}} \\
& y-y_{1}=\mathrm{bt} \longrightarrow \mathrm{t}=\frac{y-y_{1}}{\mathrm{~b}}
\end{aligned}
$$

$\therefore$ by eliminating the parameter ( t ) we obtain $\frac{x-x_{1}}{\mathrm{a}}=\frac{y-y_{1}}{\mathrm{~b}}$ which is called the cartisian or symmetric equation of $L$.

Notice that this equation can be put in the slope - point form as follows :

$$
\left(y-y_{1}\right) \mathrm{a}=\left(x-x_{1}\right) \mathrm{b} \longrightarrow y-y_{1}=\frac{\mathrm{b}}{\mathrm{a}}\left(x-x_{1}\right) .
$$

Compairing this with the slope - point form $y-y_{1}=\mathbf{M}\left(x-x_{1}\right)$ we deduce that $\mathbf{M}=\frac{b}{a}$
$\therefore$ If $\vec{V}=[\mathrm{a}, \mathrm{b}]$ is the direction vectors of $\mathbf{L}$ then the slope of $\mathbf{L}$ is $\mathbf{M}=\frac{\mathrm{b}}{\mathrm{a}}$ . This relationship can be used to deal with the two approaches ( analytic and vector ) at the same time by using this formula $\mathbf{M}=\frac{b}{a}$ to transfare from one system to anther.

## Example :

1) Find the equations the of line through $\mathbf{A}(3,-4)$ and parallel to the vectors $\vec{V}=[-4,1]$. See the figure.(6.38)
Solution : refering to the figure (6.38), we notice that $\overrightarrow{A B} / / \vec{V}$ then
$\overrightarrow{A B}=t \vec{v} \quad \vec{b}-\vec{a}=\mathrm{t}[-4,1]$
$\therefore\left[x-x_{1}, y-y_{1}\right]=\mathrm{t}[-4,1]$


Fig.(6.38)
Vector equation
$\begin{aligned} \therefore x-3 & =-4 \mathrm{t} \longrightarrow x=3-4 \mathrm{t} \\ y+4 & =\mathrm{t}\end{aligned} \quad \begin{gathered}\text { parametric } \\ \text { equations }\end{gathered}$
$\therefore \frac{x-3}{-4}=\frac{y+4}{1} \longrightarrow \frac{x-3}{-4}=y+4$
Is the symmetric ( cartesion ) equation of $L$ simplifying we get :

$$
\begin{aligned}
& -4(y+4)=x-3 \\
& -4 y-16=x+3 \longrightarrow-x-4 y-13=0 \text { or } x+4 y+13=0 .
\end{aligned}
$$

Notice that we can use the analytic method by finding the slope using $\mathrm{m}=$ $\frac{\mathrm{b}}{\mathrm{a}}$ and applying the slope - point form as :
$\therefore \mathrm{m}=\frac{1}{-4} \quad \therefore y-y_{1}=-\frac{1}{4}\left(x-x_{1}\right)$
$\therefore y+4=-\frac{1}{4}(x-3) \longrightarrow$ the equation is $\frac{x-3}{-4}=y+4$ which is equivelant to the above equation using vectors.
11) in case of the line in three - dimension (inspace) if $\vec{V}=[\mathrm{a}, \mathrm{b}, \mathrm{c}]$ is the direction vector of $\mathbf{L}$, and $\mathbf{P}\left(x_{1}, y_{1}, z_{1}\right)$ is a point on $\mathbf{L}$, ( see the figure (6.39), then:

$$
\overrightarrow{P Q}=t \vec{v}
$$

In asimilar way we can obtain the symmetric (cartesian equation) of $\mathbf{L}$ as $: \frac{x-x_{1}}{\mathrm{a}}=\frac{y-y_{1}}{\mathrm{~b}}=\frac{z-z_{1}}{\mathrm{c}}$


Fig.(6.39)
Notice that in three - dimensions the slope of the line is undefined and notice also if you are asked to find equations of the line it means that the
three types ( vector - parametric and symmetric ), which by the equation of the line they mean the symmetric only.

Example : find equations of the line through $\mathbf{P}(1,-2,3)$ and parrallel to the line through $\mathbf{A}(2,2,-3)$ and $\mathbf{B}(4,5,-2)$.

Solution : refering to the figure (6.40) since $\mathbf{L}_{\mathbf{1}} / / \mathbf{L}_{\mathbf{2}}$ then we can see that the vector $\vec{V}=\overrightarrow{A B}$ is the direction vector of $\mathbf{L}_{\mathbf{1}}$ and $\mathbf{L}_{\mathbf{2}}$

$$
\left.\begin{array}{l}
\therefore \overrightarrow{P Q}=\mathrm{t} \vec{v} \longrightarrow \vec{q}-\vec{p}=\mathrm{t}(\vec{b}-\vec{a}) \cdot \\
\therefore[x-1, y+2, \mathrm{z}-3]=\mathrm{t}[2,3,1]
\end{array}\right\} \quad \begin{aligned}
& \text { vector } \\
& \text { equations }
\end{aligned}
$$



Fig.(6.40)
$\left.\begin{array}{l}\therefore x-1=2 \mathrm{t} \longrightarrow x=1+2 \mathrm{t} \\ y+2=3 \mathrm{t} \longrightarrow y=-2+3 \mathrm{t} \\ z-3=\mathrm{t} \longrightarrow z=3+\mathrm{t}\end{array}\right\} \quad \begin{gathered}\text { parametric } \\ \text { equations }\end{gathered}$
$\therefore \frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{1}$ symmetric equation.
12) The angle between two lines:

From the figure.(6.41) we can see that the angle $(\boldsymbol{\theta})$ between the two lines $\mathbf{L}_{1}$ and $\mathbf{L}_{2}$ is the angle between their direction vectors $\overrightarrow{V 1}$ and $\overrightarrow{V 2}$, we can obtain $\theta$ by using the formula $\cos \theta=\frac{\overrightarrow{\mathrm{V} 1} \cdot \overrightarrow{\mathrm{~V} 2}}{|\overrightarrow{\mathrm{~V} 1}||\overrightarrow{\mathrm{V} 2}|}$.


Fig.(6.41)
If $\overrightarrow{V 1}$ and $\overrightarrow{V 2}$ are non zero vectors we deduce that :

$$
\begin{aligned}
& \overrightarrow{V 1} \cdot \overrightarrow{V 2}=0 \quad \mathrm{~L}_{1} \perp \mathrm{~L}_{2} \\
& \overrightarrow{V 1} \times \overrightarrow{V 2}=0 \longrightarrow \mathrm{~L}_{1} / / \mathrm{L}_{2} \\
& \overrightarrow{V 1}=\mathrm{k} \overrightarrow{V 2} \longrightarrow \mathrm{~L}_{1} / / \mathrm{L}_{2}
\end{aligned}
$$

Example : show that the line through the two points $\mathrm{A}(7,5), \mathrm{B}(1,1)$ is perpendicular to the line through the two points $C(4,-3), \mathrm{D}(2,0)$

Solution : refering to the figure.(6.42) it appears that the direction vectors of lines $\mathbf{L}_{\mathbf{1}}$ and $\mathbf{L}_{\mathbf{2}}$ are $\overrightarrow{A B}$ and $\overrightarrow{C D}$ respectevily, then


Fig.(6.42)
$\overrightarrow{A B}=\vec{b}-\vec{a}=[1,1]-[7,5]=[-6,-4]$
$\overrightarrow{C D}=\vec{d}-\vec{c}=[2,0]-[4,-3]=[-2,3]$
$\therefore \overrightarrow{A B} \cdot \overrightarrow{C D}=[-6,-4] \cdot[-2,3]=12-12=0$
$\therefore \overrightarrow{A B} \perp \overrightarrow{C D} \longrightarrow \mathrm{~L}_{1} \perp \mathrm{~L}_{2}$.
13) Equation of a plane :
I) The plane, may be completely specified by means of a non- zero. (see the figure.(6.43)).


Fig.(6.43)
Normal vector which is perpendicular to the plane and a point on the plane.
Let $\vec{N}=[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ be the normal vector and $\mathbf{P}\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}, z_{1}\right)$ be given point on the plane and $\mathbf{Q}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be an arbitrary ( general ) point on the plane. Refering to Fig.(6.43) we notice that $\vec{N}$ is perpendicular to $\overrightarrow{P Q}$.
$\vec{N} \cdot \overrightarrow{P Q}=0 \longrightarrow[\mathrm{a}, \mathrm{b}, \mathrm{c}] \cdot\left[x-x_{1}, y-y_{1}, z-z_{1}\right]=0$
Which is called the vector equation of the plane
$\therefore \mathrm{a}\left(x-x_{1}\right)+\mathrm{b}\left(y-y_{1}\right)+\mathrm{c}\left(z-z_{1}\right)=0$
Lumping the constant terms this can be written as : $a x+b y+c z+d=0$

Where $\mathbf{d}=\mathbf{a} \boldsymbol{x}_{\mathbf{1}}+\mathbf{b} \boldsymbol{y}_{\mathbf{1}}+\mathbf{c} \boldsymbol{z}_{1}$ which is called the general linear equation in three variables which represent a plane with a noraml vector $\vec{N}=[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.
II) The equation of the plane through the three points $\mathbf{A}\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}, z_{1}\right)$, $\mathbf{B}\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}, z_{2}\right)$ and $\mathbf{C}\left(\boldsymbol{x}_{3}, \boldsymbol{y}_{3}, z_{3}\right)$ can be obtained as follows :
Let $\mathbf{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be a geneal point in the plane.as shown in the figure.(6.44).


Fig.(6.44)
$\vec{N}=\overrightarrow{A B} \times \overrightarrow{A C}$ is the normal vector
$\therefore \vec{N} \perp \overrightarrow{A P}$ then

$$
\vec{N} \cdot \overrightarrow{A P}=0 \text {, or } \overrightarrow{A P} \cdot \vec{N}=0
$$

$\therefore \overrightarrow{A P} \cdot(\overrightarrow{A B} \times \overrightarrow{A C})=0$ (the vector equation).
$\therefore\left|\begin{array}{lll}x-x_{1} & y-y_{1} & z-z_{1} \\ x_{2}-x_{1} & y_{2}-y_{1} & z_{2}-z_{1} \\ x_{3}-x_{1} & y_{3}-y_{1} & z_{3}-z_{1}\end{array}\right|=0$

Example: Find an equation of the plane through the three point $\mathbf{A}(1,2,3)$, B ( $-1,1,1$ ) and $\mathbf{C}(3,-1,2)$.

## Solution:

$\overrightarrow{A P}=\vec{P}-\vec{a}=[x-1, y-2, z-3]$
$\overrightarrow{A B}=\vec{b}-\vec{a}=[-2,-1,-2]$
$\overrightarrow{A C}=\vec{c}-\vec{a}=[2,-3,-1]$

$$
\therefore \overrightarrow{\mathrm{AP}} \cdot(\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}})=\left|\begin{array}{ccc}
x-1 & y-2 & z-3 \\
-2 & -1 & -2 \\
2 & -3 & -1
\end{array}\right|=0
$$

Evaluating the determinant we get
$-5(x-1)-6(y-2)+8(z-3)=0$
Simplifying the equation we get
$-5 x-6 y+8 z-7=0$, multiplying by -1 we get
$5 x+6 y-8 z+7=0$ which represents a plane with normal vector $\vec{N}=[5,6,-8]$.
14) Show that $|\vec{A} \cdot(\vec{B} \times \vec{C})|$ is equal to the volume of a parallel piped with sides $\vec{A}, \vec{B}$ and $\vec{C}$. see the figure (6.45).


Fig.(6.45)

Proof: Associate vectors $\vec{A}, \vec{B}, \vec{C}$ with the sides of the parallelpiped and $\vec{n}$ be a unit vector normal to the base in the direction of ( $\vec{B} \times \vec{C}$ ) and let (h) be the height.
$\therefore$ volume of parallelepiped $=\mathrm{h}$ ( area of the base and
$\therefore \mathrm{h}=|\vec{A}| \cos \theta \longrightarrow \mathrm{h}=\vec{A} \cdot \vec{n}$
$\therefore$ Area of the base ( parallelogram with sides $\vec{A}$ and $\vec{C}=|\vec{B} \times \vec{C}|$
Then volume of the parallelopiped $(\vec{A} \cdot \vec{n})|\vec{B} \times \vec{C}|$
$\therefore$ Volume $=\vec{A} \cdot(|\vec{B} \times \vec{C}| \vec{n})$

$$
=\vec{A} \cdot(\vec{B} \times \vec{C})
$$

- Notice that if $\vec{A}, \vec{B}, \vec{C}$ don't form aright system then $\vec{A} \cdot \vec{n}<\overrightarrow{0}$ then volume $=|\vec{A} \cdot(\vec{B} \times \vec{C})|$.
- In similar way we can obtain the formula of the volume of the telrahedron as :
The volume $=\frac{1}{6}|\vec{A} \cdot(\vec{B} \times \vec{C})|$.

15) Proof the theoren which states that:
" the diagonals of the rhombus are perpendicular. (see the figure (6.46)).


Fig.(6.46)

Proof: let A, B, C, D be the vertices of the given rhombus . notice that the rhombus is a parallelogram with all sides equal associate vector $\overrightarrow{V 1}, \overrightarrow{V 2}$ with two adjacent sides of this rhombus thus $|\overrightarrow{V 1}|=|\overrightarrow{V 2}|$
$\therefore$ The first diagonal vector is $\overrightarrow{A C}=\overrightarrow{V 1}+\overrightarrow{V 2}$.
The second diagonal vector $\overrightarrow{B D}=\overrightarrow{V 2}-\overrightarrow{V 1}$
$\therefore \overrightarrow{A C} \cdot \overrightarrow{B D}=(\overrightarrow{V 1}+\overrightarrow{V 2}) \cdot(\overrightarrow{V 2}-\overrightarrow{V 1})$

$$
\begin{aligned}
&=\overrightarrow{V 1} \cdot \overrightarrow{V 2}-\overrightarrow{V 1} \cdot \overrightarrow{V 1}+\overrightarrow{V 2} \cdot \overrightarrow{V 2}-\overrightarrow{V 2} \cdot \overrightarrow{V 1}_{1} \\
&=\overrightarrow{V 2} \cdot \overrightarrow{V 2}-\overrightarrow{V 1} \cdot \overrightarrow{V 1}=|V 2|^{2}|V 1|^{2} \\
& \therefore|\overrightarrow{V 1}|=|\overrightarrow{V 2}| \longrightarrow \overrightarrow{A C} \cdot \overrightarrow{B D}=0 \longrightarrow \overrightarrow{A C} \perp \overrightarrow{B D}
\end{aligned}
$$

16) Derive the law of cosines for plane triangles

Proof: Associate vectors $\vec{A}, \vec{B}$ and $\vec{C}$ with the sides of the given triangle where $\boldsymbol{\theta}$ is the angle between $\vec{A}$ and $\vec{B}$. Derive the law of cosines for plane triangles, as shown in the figure (6.47)


Fig.(6.47)

Refering to Fig.(6.35): $\vec{B}+\vec{C}=\vec{A} \longrightarrow \vec{C}=\vec{A}-\vec{B}$
Then: $\vec{C} \cdot \vec{C}=(\vec{A}-\vec{B}) \cdot(\vec{A}-\vec{B})$

$$
=\vec{A} \cdot \vec{A}-\vec{A} \cdot \vec{B}-\vec{B} \cdot \vec{A}+\vec{B} \cdot \vec{B}
$$

$$
=\vec{A} \cdot \vec{A}+\vec{B} \cdot \vec{B}-2 \vec{A} \cdot \vec{B}
$$

$$
|\vec{C}|^{2}=|\vec{A}|^{2}+|\vec{B}|^{2}-2|\vec{A}||\vec{B}| \cos \theta
$$

Which can be written as :
$C^{2}=A^{2}+B^{2}-2 A B \cos \theta$
17) Derive the law of sines for the plane triangles
refering to the figure.(6.48) we have :
$\vec{C}=\vec{A}-\vec{B}$, then: $\quad \vec{C} \times \vec{C}=\vec{C} \times(\vec{A}-\vec{B})$.
Then: $\vec{O}=\vec{C} \times \vec{A}-\vec{C} \times \vec{B} \longrightarrow \vec{C} \times \vec{A}=\vec{C} \times \vec{B}$
$|\vec{C}||\vec{A}| \sin \beta=|\vec{C}||\vec{B}| \sin \propto \longrightarrow \frac{|\overrightarrow{\mathrm{A}}|}{\sin \propto}=\frac{|\overrightarrow{\mathrm{B}}|}{\sin \beta}$
To give $\frac{A}{\sin \propto}=\frac{B}{\sin \beta}=\frac{C}{\sin \gamma}$


Fig.(6.48)

Dervie the compound angle formulae :
i) $\quad \operatorname{Cos}(\mathrm{A}-\mathrm{B})=\cos \mathrm{A} \cos \mathrm{B}+\sin \mathrm{A} \sin \mathrm{B}$, and
ii) $\quad \operatorname{Sin}(\mathrm{A}-\mathrm{B})=\sin \mathrm{A} \cos \mathrm{B}-\sin \mathrm{B} \cos \mathrm{A}$

Let $\overrightarrow{u 1}, \overrightarrow{u 2}$ be two unit vectors such that (see the figure.(6.49))
$\overrightarrow{U 1}=[\cos \mathrm{A}, \sin \mathrm{A}]$
$\overrightarrow{U 2}=[\cos \mathrm{B}, \sin \mathrm{B}]$ Then:
i) $\cdot \overrightarrow{u 1} \cdot \overrightarrow{u 2}=|\overrightarrow{u 1}||\overrightarrow{u 2}| \cos (A-B)$, since $\left|\mathrm{u}_{1}\right|\left|\mathrm{u}_{2}\right|=1$
$\therefore[\cos A, \sin A] .[\cos B, \sin B]=\cos (A-B)$
$\therefore \cos (A-B)=\cos A \cos B+\sin A \sin B$.
ii) $\overrightarrow{u 1} \times \overrightarrow{u 2}=\left|\begin{array}{ccc}\vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos A & \sin A & 0 \\ \cos B & \sin B & 0\end{array}\right|$

$$
=(\sin \mathrm{A} \cos \mathrm{~B}-\sin \mathrm{B} \cos \mathrm{~A}) \vec{k}
$$

$\therefore \overrightarrow{u 1} \times \overrightarrow{u 2}=|\overrightarrow{u 1}||\overrightarrow{u 2}| \cos (A-B) \vec{k}=\cos (A-B) \vec{k}$
Then: $\cos (\mathrm{A}-\mathrm{B}) \vec{k}=(\sin \mathrm{A} \cos \mathrm{B}-\sin \mathrm{B} \cos \mathrm{A}) \vec{k}$
$\therefore \cos (A-B)=\sin A \cos B-\sin B \cos A$.


Fig.(6.49)

## Unit .7-Calculus concepts.

7.1 - Function.
7.1.1 - Function in school mathematics.
7.1.2 - Definition of function.
7.1.3 - Recommendations.
7.1.4 - Inverse function.
7.1.5 - Types of functions.
7.1.6 - Composite function.
7.1.7 - Algebra of functions.
7.2 - Limit of a function.
7.2.1 - The formal definition of limit.
7.2.2 - Types of limits.
7.2.3 - Basic limit theorems.
7.2.4 - Continuous function.
7.3 - Derivative of function.
7.3.1 - The formal definition of the derivative.
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7.4.1 - Properties of definite integral.
7.4.2 - The indefinite integral.
7.4.3 - Table of commonly used integration formulas.

## Introduction

The word " calculus " is derived from a latin word meaning " stone " or " pebble ", which was used in doing sums and counting, calculus eventually came to mean a method of calculation.

The calculus is a mathematical process invented in the seventeenth $\left(17^{\text {th }}\right)$ century by Newton and Leibniz (1646-1716).

They are regarded as the founders of calculus because they were the first to understand the relation between differentiation and integration and also to exploit this relation by applying the second form of the fundamental theorem of calculus. they were the ones who systematized the calculus into an organized body of mathematical knowledge.

Calculus applications caused a revolution in the scientific development. calculus is also the basis of another branch of mathematics called analysis, and it is considered to be the outstanding intellectual achievement of the human race.

The subject called calculus is built up on three basic concepts, the concepts of a variable the concepts of a function, and the concept of a limit.

The two branches of calculus are concerned with processes on the function, the differential calculus is concerned with the process of finding the derivative of a function, this operation is called differentiation.

The integral calculus is concerned with the process of finding the integral of a function, this operation is called integration.

In this unit we will introduce the main concepts related to calculus pointing out their terminology and definitions and some examples associated with them.

### 7.1.Function

The concept of a function is one of the basic concepts of mathematics and is particularly important in the application of mathematics in the real world.
historically the concept of function first introduced by Leibniz (16461716), Euler (1707-1783) was the first to use the expression $\mathrm{f}(x)$. the idea
of function was linked to the idea of dependence. this indicates that the concept of function grew in order to express mathematically the notion of dependence of one quantity on another, in which the second quantity is determined from the first quantity by using some rule, then the second quantity was called a dependent quantity, and the first was called an independent quantity, sometimes the rule of dependence can be expressed in a form of an algebraic formula such as $A=\pi r^{2}$ which indicates that the area $(A)$ of circle depends up on the radius $(r)$, in the case of variables when we write a formula as $y=a x+b$ where a and d are real numbers we call the variable $(x)$ as the independent or (first ) variable and the variable $(y)$ as the dependent or ( second ) variable.

This means that as $(x)$ changes $(y)$ will change according to that.
This means that each change of $(x)$ is followed by a change in $(y)$ which means that the change in $(y)$ is a sign or a result of a change in $(x)$ then $(y)$ is called a function of $(x)$ and written algebraically as $Y=f(x)$. the idea of a function has gradually been refined and clarified over the years and passed through many stages of development from the idea of dependence to the idea of association and correspondence and has only fairly recently been seen in its full generality.

### 7.1.1. Function in school mathematics.

Different approaches of introducing the concept of function were followed by different school mathematics textbooks, one of the most convenient approaches to young pupils was found to be the one which follows the stepping from the whole to the parts. this approach introduces the idea of a relation through a set of concrete examples which describe real life situations which are important in building up a knowledge of the common functions these examples describe the different degrees and types of association and correspondence (one to one - many to one - one to many many to many).
From early stages in the primary school pupils should study the different ideas and ways of representation of relations through concrete examples showing the different types and degree of association and correspondence the emphasize should be put on the idea of verbal description of the rule and the idea of arrow diagram representation. Because it is very simple and
comprehensible to almost all pupils and gives a good picture of the situation. In another stage the pupils can identify a special type of relations with an additional feature and property, these relations are then called functional relations or simply (functions), they are those of one to one and many to one correspondence.

This approach considered the function as a particular type of relation that is the concept of relation is more general than that of function, which means that the set of functions is a subset of the set of relations, as shown in the figure.(7.1).

i.e $\quad \mathrm{F} \subset \mathrm{R}$

Fig.(7.1).

### 7.1.2. definition of function:

The previous ideas and arguments suggest the following definitions.
1- A relation from set $\boldsymbol{A}$ to set $\boldsymbol{B}$ is a connection or correspondence between the elements of these two sets.
2- A function from set $\boldsymbol{A}$ to set $\boldsymbol{B}$ is a relation between these two sets such that each element in $\boldsymbol{A}$ relates (corresponds) to one and only one element in $\boldsymbol{B}$.
3- Contemporary definition of function.
Taking in account the idea of dependence and association modern texts regard a function as having three parts or components, and use these to formulate a more general definition of function as;
I) A starting set $\boldsymbol{A}$ called the domain of the function whose all elements are processed.
II) A target set $\boldsymbol{B}$ called the co- domain of the function a single element of which is related to each element of the domain (A).
III) An association rule or process which assigns to each element in the starting set $\boldsymbol{A}$, a single element (i.e. only one element) of the target set B . then the process is called a function and denoted by the letter $(f)$ and written as $f: \boldsymbol{A} \rightarrow \boldsymbol{B}$ and shown diagrammatically, as shown in the figure.(7.1).


Fig.(7.2).
IV) Formal definition of function:

In recent years some mathematicians concerned with the logical and conceptual foundations of their subject suggested an axiomatic development of ideas and topics in such an order of presentation leeds to a formal definition of function as follows:
i) The idea of a set is first studied.
ii) The idea that two elements (a) and (b) can be put together to form an ordered pair ( $\mathbf{a}, \mathbf{b}$ ) is introduced.
iii) The Cartesian product $\mathbf{A} \times \mathbf{B}$ of two sets $\mathbf{A}$ and $\mathbf{B}$ is defined as a set of ordered pairs ( $\mathbf{a}, \mathbf{b}$ ) such that $\mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}$.
iv) A relation from $\mathbf{A}$ to $\mathbf{B}$ is defined to be a subset of $\mathrm{A} \times \mathrm{B}$
v) A function $\boldsymbol{f}$ with domain $\mathbf{A}$ and co domain $\mathbf{B}$ is now a relation from $\mathbf{A}$ to $\mathbf{B}$ with an additional property that each $\mathbf{a} \in \mathbf{A}$ is the first member of exactly one ordered pair belonging to the function.
We notice that this definition considers the function as a particular (special) type of relation.
It can be seen also that the formal definition of function represents a formal and axiomatic approach of introducing this concept which is above the level of thinking of the pupils in the age range ( $7-15$ years) being still in the stage of concrete operational thinking according to piaget's theory of cognitive development.

### 7.1.3.Recommendations

We will suggest here a view point about introducing the concept of function in a sequence of stages showing the order of development of ideas and topics taking in account the spiral approach of presentation and mathematical and psychological and Educational aspects associated with stage of the cognitive development of the pupils concerned.

These stages start from early school life of the pupils it could be from nursery through primary, preparatory and secondary school during these stages children build up the foundations of mathematical concepts through every day activities and using the environment as a source of producing concert materials and tools of learning relationally through experiences and observations in the real word.

## Stage . 1 : Introducing

The ideas of classification, ordering, arranging, sorting, comparing, representing and recognizing patterns, trough playing with actual and real objects such as beads - dolls - glasses - bottles - cubes - blocks - small balls ... etc.

## Stage . 2 : Introducing

The idea of sets and their relations such as belonging, number of elements - in every simple form without using any symbols or terms, Venn diagrams play a good rule in this aspect.

## Stage . 3 Introducing

The idea of association and relating objects to each other such as one to one correspondence through playing with real objects, which is used as a base for number concept and counting.

## Stage . 4 Introducing

The idea of relation between sets of real objects through examples of real situations using the arrow diagram representation to show various types of association and correspondence the relation now can be defined simply as a connection between two sets.

## Stage . 5 introducing

The concept of a set and ways of representation of sets and operations on sets using symbolization and terms through concrete examples and real situation leading to the formal representations.

Stage . 6 Introducing
The concept of function in a simple form as a particular type of relation. The function from set A to set B is now defined as a relation from A to B such that each element in A relates (corresponds) to one and only one element in B introducing ways of function representation and types of functions and algebra of functions and composition of functions.

## Stage . 7 Introducing

The ideas of ordered pairs - the Cartesian product of two sets definition of a relation from $\mathbf{A}$ to $\mathbf{B}$ as a subset of $\mathbf{A} \times \mathbf{B}$ definition of a function from $\mathbf{A}$ to $\mathbf{B}$ as a relation from $\mathbf{A}$ to $\mathbf{B}$ with an additional property that each element $\mathbf{a} \in \mathbf{A}$ is the first member of exactly one ordered pair of the function.

### 7.1.4.Inverse function:

The idea of the inverse function appears through the idea of reversing the procedure of association of this function as shown in the figure(7.3).


Fig.(7.3)

## Notes:

i) If the function f is one -to- one then the inverse of $\boldsymbol{f}$ is a function and denoted by $\boldsymbol{f}^{-\mathbf{1}}$
ii) The inverse of a function $f$ may not be a function
iii) Domain of $\boldsymbol{f}^{-\mathbf{1}}=$ range of $\boldsymbol{f}$, range of $\boldsymbol{f}^{-\mathbf{1}}=$ domain of $\boldsymbol{f}$.
iv) A method that generates a formula for $\boldsymbol{f}^{\mathbf{- 1}}$ is as follows:
a) Write $y=f(x)$.
b) interchange $x$ and $y$ or solve for $x$ in terms of $y$.
c) write $f^{-1}(x)=y$ directly or $f^{-1}(y)=x$ in terms of $y$ and then interchange $x$ and $y$.

## Examples:

1) Specify $\boldsymbol{f}^{\mathbf{- 1}}$ given
$\boldsymbol{f}=\{(1,2),(3,5),(6,4),(2,0)\}$
Solution: being f is one - to - one then $\boldsymbol{f}^{\mathbf{- 1}}$ is a function, in this type of representation we use arrow diagram and interchange the first and second components in each of the ordered pairs as shown in the figure (7.4).


Fig.(7.4)
$\therefore f^{-1}=\{(2,1),(5,3),(4,6),(0,2)\}$
2) Find $f^{-1}$ given $f(x)=3 x-2$.

Solution: following the steps listed to obtain:

$$
\begin{aligned}
& y=3 x-2 \\
& x=\frac{y+2}{3} \\
& \therefore f^{-1}(y)=\frac{y+2}{3}
\end{aligned}
$$

$$
\text { or } \quad y=3 x-2
$$

$$
x=3 y-2
$$

Write $f^{-1}(x)=\frac{x+2}{3}$

### 7.1.5- Types of functions:

The name of function differs according to the rule of association and formula describing the relation between the variables in this function such as:
1- Algebraic functions: are those functions whose formula for $f(x)$ can be obtained by performing a finite number of operations of addition, subtraction, multiplication, division, and root extraction upon $x$ and constants such as: $f(x)=x^{2}-3 x+5$

## 2- Polynomial functions:

Is a function specified by the equation:

$$
f(x)=a_{0} x^{n}+a_{1} x^{n-1}+a_{2} x^{n-2}+\cdots+a_{n-1} x+a_{n}
$$

where $a_{0}, a_{1}, a_{2}, \ldots, a_{\text {n }}$
are real numbers and $a_{0} \neq 0$ and $n$ is a non-negative integer called the degree of the polynomial.
Special kinds of polynomial function are:
i) The first - degree function: is a function specified by the equation $y=a x+b$ or $f(x)=a x+b$, where $a, b$ are real numbers and $a \neq 0$
Example $f(x)=2 x+3$
ii) The identity function is a special case of the first - degree function and specified by the equation $f(x)=x$, or $y=x$
iii) The constant function:

Is a function specified by the equation $f(x)=c$ or $y=c$, where $c$ is a real number
Example: $f(x)=5, f(x)=-3$
iv) The second degree function:

Which is also called a quadratic function is a function specified by the equation $f(x)=a x^{2}+b x+c$ or $\quad y=a x^{2}+b x+c$ where $a, b$, and $c$ are real numbers and $\mathrm{a} \neq 0$.

## Examples:

i) $\quad f(x)=3 x^{2}-2 x+5$
ii) $\quad f(x)=x^{2}$ is a simple second - degree function.

## 3 - Rational function:

Is a function defined by the quotient of two polynomials, that is if $\boldsymbol{F}_{\boldsymbol{1}}, \boldsymbol{F}_{2}$ are two polynomial functions of $x$, then the function f specified by

$$
F(x)=\frac{f_{1}(x)}{f_{2}(x)} \text { is a ration function. }
$$

Example: $f(x)=\frac{2 x^{2}-3 x+1}{x-3}$

## 4 - Root function

Is a function for which a formula for $f(x)$ contains an nth root.

## Examples:

i) $\quad F(x)=\sqrt{x}$ is a simple square root.
ii) $\quad F(x)=\sqrt[3]{x}$ is a simple cube root.
iii) $\quad F(x)=\left(x^{2}-1\right) \frac{2}{3}+3 x^{4}-2$

## 5 - Absolute value function:

Is a function specified by the equation $f(x)=|x|$ or $y=|x|$.

## 6- Greatest integer function:

Is a function specified by the equation $f(x)=[x]$, or $y=[x]$ where each $x$ corresponds to the greatest integer less than or equal to $x$.
Examples: $f(x)=[x]=\quad\left\{\begin{array}{rc}-3 & \text { for }-3 \leq x<-2 \\ -2 & \text { for }-2 \leq x<-1 \\ -1 & \text { for }-1 \leq x<0 \\ 0 & \text { for } 0 \leq x<1 \\ 1 & \text { for } 1 \leq x<2 \\ 2 & \text { for } 2 \leq x<3 \\ 3 & \text { for } 3 \leq x \leq 4\end{array}\right.$

## 7-Picewise function

Is a function specified by more than one formula which together define a single function.

## Example:

$$
f(x)=\left\{\begin{array}{rcc}
-2 & \text { if } & -4 \leq x<-2 \\
x & \text { if } & -2 \leq x<4 \\
4 & \text { if } & 4 \leq x<6
\end{array}\right.
$$

## Notes:

i) Every polynomial function is a rational function but not every rational function is a polynomial function.
ii) Every polynomial function is an algebraic function but not every algebraic function is a polynomial function.
iii) Transcend dental functions:

They are non-algebraic functions. these include the following functions:
1- Trigonometric functions:
Trigonometric function is a function specified by an equation in terms of a trigonometric ratio of an angle, such as:
$y=f(x)=\sin x, \quad y=f(x)=\cos x$
$y=f(x)=\tan x, \quad y=f(x)=\cot x$
$y=f(x)=\sec x, \quad y=f(x)=\operatorname{cosec} x$
2- Inverse trigonometric functions are functions specified by an equation with respect to the inverse of one of the previous trigonometric functions such as:
i) $y=\sin ^{-1} x$

$$
=\arcsin x, y \in\left[-\frac{\pi}{2} ; \frac{\pi}{2}\right]
$$

ii) $y=\cos ^{-1} x$

$$
\begin{aligned}
& \Longleftrightarrow x=\cos y \\
& =\arccos x, y \in[0 ; \pi]
\end{aligned}
$$

iii) $y=\tan ^{-1} x \Longleftrightarrow x=\tan y$

$$
=\arctan x, y \in\left(\frac{-\pi}{2} ; \frac{\pi}{2}\right)
$$

iv) $y=\cot ^{-1} x \Longleftrightarrow x=\cot y$

$$
=\operatorname{arccot} x, y \in(0 ; \pi)
$$

v) $y=\sec ^{-1} x \longleftrightarrow x=\sec y$

$$
=\operatorname{arcsec} x, y \in\left[-\pi ;-\frac{\pi}{2}\right) \cup\left[0 ; \frac{\pi}{2}\right)
$$

vi) $\quad Y=\csc ^{-1} x \quad \Longleftrightarrow x=\operatorname{cosec} y$ $=\operatorname{arccosec} x \quad, y \in\left(-\infty ; \frac{-\pi}{2}\right) \cup\left(0 ; \frac{\pi}{2}\right]$

## 3- Exponential function:

Is a function specified by the equation $f(x)=a^{x}$, or $y=a^{x}$
Where $x$ is a real number called the index and $a \neq 1$ is a positive real number called the base.
Example: $y=2^{x}, y=\left(\frac{1}{3}\right)^{x}$
A special case is $y=e^{x}$ where $e=2.7182818$ which adapted by (Euler, 1707, 1783).
4- Logarithmic function:
Is the inverse of the exponential function which is specified with the equation $f(x)=\log _{a} x$ or $y=\log _{a} x$
Which is read as logarithm of $x$ to the base $a$, and as a result to the relation $y=\log _{a} x \longleftrightarrow x=a^{y}$

Example: $y=\log _{2} x, \quad y=\log _{\frac{1}{2}} x$

$$
y=\log _{10} x
$$

A special case is $\boldsymbol{y}=\log _{\boldsymbol{e}} \boldsymbol{x}$ and written in short form as $y=\ln x$ and is called natural logarithm.

### 7.1.6. Composite function:

There are different ways of combining functions one of them is the composition of two functions which is a method of constructing a new function from two given functions, the new function is then called the composite function or the function of function or the resultant function which is defined as follows:

If $\boldsymbol{f}$ and $g$ are two functions applied in two separate stages to the elements of three sets, ( $\mathrm{X}, \mathrm{Y}$ and Z ) and can be replaced by a single function acting directly from the first set $(X)$ to the last set $(Z)$ then the resulting function is called the composition of these two functions .

The following diagram (figure (7.5)) shows the process of this composition:

```
If f:X }\longrightarrow\textrm{X},g:
```



Are two functions acting separately as shown then:


Fig.(7.5)
more generally if f and g are two functions then according to the figure the composite function of f and g is defined by the rule:

```
(gof)(x)=g[f(x)] and (fog)(x)=f[g(x)]
```


## Notes

i) The expression $g \circ f$ is read as " $g \circ f$ " $f$ or $g$ composite with $f$ or " $g$ circle $f^{\prime \prime}$
ii) The composite function $g \circ f$ or $g[f]$ is a result of performing $f$ first and then performing $g$ secondall, this means that the expression $f 0 g$ indicates that the function $g$ performed firstly and $f$ performed secondally.
iii) The functions $g \circ f$ and $f \circ g$ may not be the same which means $g$ of $\neq f$ o $g$.
iv) If $I$ is the identity function then $I$ o $f=f \circ I=f$.

## Examples:

1) Determine ( find ) the functions fog and gof where :

$$
\begin{aligned}
& f=\{(0,5),(1,2),(3,1),(4,3),(5,6),(6,-3),(7,4),(8,-5)\} \\
& g=\{(2,0),(5,4),(1,-2),(3,3),(6,1),(-3,-5),(4,4),(-5,-1)\}
\end{aligned}
$$

solution: being the functions $f$ and $g$ are given as sets of ordered pairs the composition can be found more easily by using arrow diagram representation as follows:

- To obtain (fog) apply $\boldsymbol{g}$ first then apply $\boldsymbol{f}$, as shown in the figure (7.6):


Fig.(7.6)
From the diagram:
$f \circ g=\{(2,5),(3,1),(6,2),(4,3)\}$
to obtain gof apply $\boldsymbol{f}$ first then apply $\boldsymbol{g}$ as shown in the figure(7.7):


Fig.(7.7)

From the diagram it can be seen that

$$
g o f=\{(0,4),(1,0),(3,-2),(4,3),(5,1),(6,-5),(7,4),(8,-1)\}
$$

2) Find $f o g$ and $g o f$ of the given function

$$
f(x)=x^{2}+3 x, g(x)=\frac{x-3}{2}
$$

## Solution:

Being the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are given in an algebraic form we donot need to know which one is applied first, so we use the rule of definition of the composite function directly as follows:

$$
\begin{aligned}
(f \circ g)(x) & =f[g(x)] \\
= & f\left[\frac{x-3}{2}\right] \\
= & \left(\frac{x-3}{2}\right)^{2}+3\left(\frac{x-3}{2}\right) \\
= & \frac{x^{2}-9 x}{4} \\
(g \circ f)(x) & =g[f(x)] \\
= & g\left[x^{2}+3 x\right] \\
= & \frac{\left(x^{2}+3 x\right)-3}{2} \\
= & \frac{x^{2}+3 x-3}{2}
\end{aligned}
$$

### 7.1.7.Algebra OF functions :

Another way of combining functions which leeds to the construction of a new function from given functions is performing an algebraic operation ( + , - , $\times, \div$ ) on two given functions, these new functions are called respectively the ( sum - differnce - product - quotiont ) of these two given functions which are defined as:

$$
\left.\begin{array}{l}
(f+g)(x)=f(x)+g(x) \\
(f-g)(x)=f(x)-g(x) \\
(f \cdot g)(x)=f(x) \cdot g(x) \\
\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}, g(x) \neq 0
\end{array}\right\} \quad x \in D_{f} \cap D_{g}
$$

## Examples:

i) Find $f+g, f-g, f . g \quad, \frac{f}{g}$ where
$f=\{(4,3),(5,6),(0,5),(3,2),(8,11\})$
$g=\{(5,-4),(0,6),(3,3),(8,9),(7,10)\}$

## solution :

in this type of representation we find $D_{f}, D_{g}$ then $D_{f} \cap D_{g}$ as follows :
$D_{f}=\{4,5,0,3,8\}$
$D_{g}=\{5,0,3,8,7\}$
$D_{f} \cap D_{g}=\{5,0,3,8\}$
Using the definitions of these operations we have :
$f+g=\{(5,2),(0,11),(3,5),(8,20)\}$
$f-g=\{(5,10),(0,-1),(3,-1),(8,2)\}$
$f . g=\{(5,-24),(0,30),(3,6),(8,99)\}$ $\frac{f}{g}=\left\{\left(5, \frac{-3}{2}\right),\left(0, \frac{5}{6}\right),\left(3, \frac{2}{3}\right),\left(8, \frac{11}{9}\right)\right\}$
ii) Find $f+g, f-g, f . g, \frac{f}{g}$ where $f(x)=\sqrt{x-2}, g(x)$

$$
=x^{2}-1
$$

Solution : we find $D_{f}, D_{g}$ and $D_{f} \cap D_{g}$ :
$\therefore f(x)=\sqrt{x-2} \Longrightarrow x-2 \geq 0 \Rightarrow x \geq 2$
$\therefore D_{f}=[2, \infty)$
$g(x)=x^{2}-1 \quad \Longrightarrow \quad D_{g}=\mathrm{R}$
$D_{f} \cap D_{g}=\mathrm{R} \cap[2, \infty)=[2, \infty)$.
$(f+g)(x)=\sqrt{x-2}+\left(x^{2}-1\right)$
$(f-g)(x)=\sqrt{x-2}-\left(x^{2}-1\right)$
$(f . g)(x)=(\sqrt{x-2})\left(x^{2}-1\right) \quad x \in(z, \infty)$
$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{x-2}}{\left(x^{2}-1\right)} ; x \neq \pm 1$

## 7. 2. limit of a function :

Formal work on differentiation intended to the higher education students will be much more intelligible if the limit idea is allowed to grow slowly
and informally before the official notation for limits is introduced, this can be done early from primary school where children meet many situations in which a process is repeated a gain and again many of these repetitive processes are concerned with sequences, pupils should investegate how terms of a sequence grow through easy situation such as:

```
(1,3,7,13\ldots),( 2, 3, 5, 8,12 ...)
(1,2,4,8,16,32,\ldots),(2,5,10,17,26, 37,\ldots)
```

It is noticed that the ideas about limits begin to develop when situations can be explored in which a sequence converges.
The notion of limit involves the way $f(\mathrm{x})$ behaves when ( x$)$ is near a particular number (a), this indicates that as ( x ) gets closer and closer to (a), $f(\mathrm{x})$ gets closer and closer to the value ( L ) we express this idea verbally by saying that $" L$ is the limit of $f(x)$ as $x$ approaches a, or tends to a " and we express this symbolically by the notation $\lim _{x \rightarrow a} f(x)=L$.

### 7.2.1.The formal definition of limit :

Let ( $f$ ) be a function defined at each point of some open interval containing (a) except possibly at (a) itself, then a number (L) is the limit of $(f)$ at (a) i.e. $\lim _{\mathrm{x} \rightarrow \mathrm{a}} f(\mathrm{x})=\mathrm{L}$

If and only if for every number $\boldsymbol{\epsilon}>\mathbf{0}$ there exists a number $\boldsymbol{\varsigma}>\mathbf{0}$ such that if $0<|\mathrm{x}-\mathrm{a}|<\varsigma$ then $|\mathrm{f}(\mathrm{x})-\mathrm{L}|<\epsilon$.

This can be represented graphically as shown in the figure(7.8) :


Fig.(7.8)

$$
\lim _{x \rightarrow a} f(x)=L
$$

Notes: if such an $\mathbf{L}$ can be found we say that the limit of $(f)$ at (a) exists or that $\lim _{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x})$ exists, or that $(f)$ has alimit at a.

### 7.2.2 .Types of limits:

Limit of a function is called according to the way of approaching such as:
i) The two - sided limits are those ordinary limits indicated previously which are written as $\lim _{x \rightarrow \mathbf{a}} f(x)$ which means that $(x)$ approaches (a) from both sides.
ii) The one - sided limits are those in which the approach of $(x)$ from one side of (a), they are the right - hand and left - hand limits as shown in the figure (7.9):


Fig.(7.9)
iii) The limits at infinity :

They are those limits when ( $x$ ) approaches ( $\infty$ or $-\infty$ ) and written as: $\lim _{x \rightarrow \infty^{+}} f(x), \lim _{x \rightarrow \infty^{-}} f(x)$.

### 7.2.3.Basic limit theorems :

We will list some theorms on limits which can be used as rules in evaluating limits, they are simpler to apply than the formal ( $\epsilon, \varsigma)$ definition of limit of a function:

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are real numbers and $\boldsymbol{f}, \mathbf{g}$ are functions in $x$ such that:
$\lim _{x \rightarrow \mathrm{a}} f(x)=\mathrm{L}_{1}$, and $\lim _{x \rightarrow \mathrm{a}} \mathrm{g}(x)=\mathrm{L}_{2}$ then:
i) $\quad \lim _{x \rightarrow \mathrm{a}} \mathrm{b}=\mathrm{b} \quad$ limit of a constant function
ii) $\quad \lim _{x \rightarrow \mathrm{a}} x=\mathrm{a} \quad$ limit of identity function
iii) $\quad \lim _{x \rightarrow \mathrm{a}} \mathrm{c} f(x)=\mathrm{c} \lim _{\mathrm{x} \rightarrow \mathrm{a}} f(x)=\mathrm{cL}_{1}$ limit of constant multiple
iv) $\lim _{x \rightarrow \mathrm{a}}[f(x)+\mathrm{g}(x)]=\lim _{x \rightarrow \mathrm{a}} f(x)+\lim _{x \rightarrow \mathrm{a}} \mathrm{g}(x)=\mathrm{L}_{1}+\mathrm{L}_{2}$ Limit of the sum
v) $\quad \lim _{x \rightarrow \mathrm{a}}[f(x)-\mathrm{g}(x)]=\lim _{x \rightarrow \mathrm{a}} f(x)-\lim _{x \rightarrow \mathrm{a}} \mathrm{g}(x)=\mathrm{L}_{1}-\mathrm{L}_{2}$

Lim of the difference
vi) $\quad \lim _{x \rightarrow \mathrm{a}}[f(x) \cdot \mathrm{g}(x)]=\lim _{x \rightarrow \mathrm{a}} f(x) \cdot \lim _{x \rightarrow \mathrm{a}} \mathrm{g}(x)=\mathrm{L}_{1} \cdot \mathrm{~L}_{2}$ limit of the product
vii) $\quad \lim _{x \rightarrow \mathrm{a}}\left[\frac{f(x)}{\mathrm{g}(x)}\right]=\frac{\lim _{x \rightarrow \mathrm{a}} f(x)}{\lim _{x \rightarrow \mathrm{a}} \mathrm{g}(x)}=\frac{L_{1}}{L_{2}} \quad, L_{2} \neq 0$

Limit of the quotient.
viii) $\lim _{x \rightarrow \mathrm{a}}[f(x)]^{n}=\left[\lim _{x \rightarrow \mathrm{a}} f(x)\right]^{\mathrm{n}}=\left[\mathrm{L}_{1}\right]^{\mathrm{n}}$ limit of the power
ix) $\lim _{x \rightarrow \mathrm{a}} \sqrt[n]{f(x)}=\sqrt[n]{\lim _{x \rightarrow \mathrm{a}} f(x)}=\lim _{x \rightarrow \mathrm{a}} \sqrt[n]{L_{1}}$

Limit of the nth root .

## Notes:

i) The first step we follow to evaluate the limit of a function is trying the direct substitution on ( $x$ ) by the number (a) in the formula ( equation ) defining this function.
If the result was a non real number such as $\frac{\mathbf{0}}{\mathbf{0}}, \frac{\infty}{\infty}, \sqrt{-\boldsymbol{v} \boldsymbol{e}}$ which are also called non defined quantities in this case we have to use some algebraic methods such as factorising or multiplying by the conjugate.
iii) when a function is defined by a formula involving more than one equation one - sided limits are often used to show that the function has a limit at a given point.
iv) The $\lim _{x-\mathrm{a}} f(x)$ exists if and only if both one - sided limits exist and equal, in this case $\lim _{x \rightarrow \mathrm{a}^{-}} f(x)=\lim _{x \rightarrow \mathrm{a}^{+}} f(x)=\lim _{x \rightarrow \mathrm{a}} f(x)$
Examples :
Evaluate ( find ) each of the following limits if it exists.

1) $\lim _{x \rightarrow 2} 3=3$
2) $\lim _{x \rightarrow 2} x=2$
3) $\lim _{x \rightarrow-1} x^{3}=(-1)^{3}=-1$
4) $\lim _{x \rightarrow-3} \frac{2}{x^{2}}=\frac{2}{(-3)^{2}}$

$$
=\frac{2}{9}
$$

5) $\lim _{x \rightarrow 1} 4 x^{2}+x^{3}+2=4(1)^{2}+(1)^{3}+2$

$$
\begin{aligned}
& =4+1+2 \\
& =7
\end{aligned}
$$

6) $\left.\lim _{x \rightarrow 4} 2 x^{3}\left(x^{2}-1\right)=2(4)^{3}\left[(4)^{2}-1\right)\right]$

$$
\begin{aligned}
& =2(64)(16-1) \\
& =128(15) \\
& =1920
\end{aligned}
$$

7) $\lim _{x \rightarrow-2} \frac{3-x^{2}}{1+x^{3}}=\frac{3-(-2)^{2}}{1+(-2)^{3}}$

$$
\begin{aligned}
& =\frac{3-4}{1-8} \\
& =\frac{-1}{-7}=\frac{1}{7}
\end{aligned}
$$

8) $\lim _{x \rightarrow 2}\left(2+x^{3}\right)^{2}=\left[2+(2)^{3}\right]^{2}$

$$
\begin{aligned}
& =(2+8)^{2} \\
& =(10)^{2}=100
\end{aligned}
$$

9) $\lim _{x \rightarrow 2} \sqrt[3]{x^{3}+x-2}=\sqrt[3]{(2)^{3}+(2)-2}$

$$
\begin{aligned}
& =\sqrt[3]{8+2-2} \\
& =\sqrt[3]{8} \\
& =2
\end{aligned}
$$

10) $\lim _{x \rightarrow 2} \frac{x^{2}+x-6}{x^{2}-4}=\lim _{x \rightarrow 2} \frac{(x+3)(x-2)}{(x-2)(x+2)}$

$$
=\lim _{x \rightarrow 2} \frac{(x+3)}{(x+2)}=\frac{5}{4}
$$

11) $\lim _{x \rightarrow 3} \frac{\sqrt{x}-\sqrt{3}}{x^{2}-3 x}=\lim _{x \rightarrow 3} \frac{(\sqrt{x}-\sqrt{3})}{x(x-3)}$

$$
\begin{aligned}
& =\lim _{x \rightarrow 3} \frac{(\sqrt{x}-\sqrt{3})}{x(x-3)} \cdot \frac{(\sqrt{x}+\sqrt{3})}{\sqrt{x}+\sqrt{3}} \\
& =\lim _{x \rightarrow 3} \frac{(x-3)}{x(x-3)(\sqrt{x}+\sqrt{3})} \\
& =\lim _{x \rightarrow 3} \frac{1}{x(\sqrt{x}+\sqrt{3})}=\frac{1}{3(\sqrt{3}+\sqrt{3})}=\frac{1}{6 \sqrt{3}} \\
& \frac{1}{x^{2}}=\lim _{x \rightarrow \infty} \frac{\frac{3 x^{2}}{x^{2}}-\frac{2 x}{x^{2}}+\frac{1}{x^{2}}}{x^{2}-\frac{3 x}{x^{2}}+\frac{6 x 2}{x^{2}}} \\
& \quad=\lim _{x \rightarrow \infty} \frac{3-\frac{2}{x}+\frac{1}{x^{2}}}{\frac{4}{x^{2}}-\frac{3}{x}+6} \\
& \quad=\frac{3}{6}=\frac{1}{2}
\end{aligned}
$$

12) $\lim _{x \rightarrow \infty} \frac{3 x^{2}-2 x+1}{4-3 x+6 x^{2}}=\lim _{x \rightarrow \infty} \frac{\frac{3 x^{2}}{x^{2}}-\frac{2 x}{x^{2}}+\frac{1}{x^{2}}}{\frac{4}{x^{2}}-\frac{3 x}{x^{2}}+\frac{6 x 2}{x^{2}}}$
13) $\lim _{x \rightarrow 0} \frac{(1-x)}{x(x-1)}=\lim _{x \rightarrow 0} \frac{-(x-1)}{x(x-1)}$

$$
=\lim _{x \rightarrow 0} \frac{-1}{x}
$$

$$
=-\frac{-1}{0} \text { does not exist ( or not defined). }
$$

14) $f(x)=\left\{\begin{array}{r}2+x ; x>2 \\ 6-x ; x \leq 2\end{array} \quad\right.$ ( see the figure 7.10)

$$
f(x)=6-x \quad f(x)=2+x
$$

2

Fig.(7.10)

$$
\begin{gathered}
\lim _{x \rightarrow 2^{+}} f(x)=\lim _{x \rightarrow 2^{+}}(2+x) \\
=2+2=4 \\
\lim _{x \rightarrow 2^{-}} f(x)=\lim _{x \rightarrow 2^{-}}(6-x) \\
=6-2=4
\end{gathered}
$$

```
\(\lim _{\mathrm{x} \rightarrow 2^{-}} f(x)=\lim _{\mathrm{x} \rightarrow 2^{+}} f(x)=4\)
\(\lim _{x \rightarrow 2} f(x)=4 \quad\) exists .
```


### 7.2.4.Continuous function :

Continuity means change with out break, this indicates that the function defined by an equation $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ is said to be continuous at $\boldsymbol{x}=\boldsymbol{a}$ if its graph is unbroken at that point, this suggests the following definition :

The function $(f)$ is continuous at $(\boldsymbol{a})$ if :
i) $\quad \boldsymbol{a}$ is in the domain of $(\boldsymbol{f})$ that is $\boldsymbol{f}(\boldsymbol{a})$ is defined, exists.
ii) $\lim _{x \rightarrow \mathrm{a}} f(x)$ exists
iii) $\lim _{x \rightarrow \mathrm{a}} f(x)=f(a)$

## Notes:

i) ( cauchy 1821 ) suggested that these three conditions can be combined into one single condition as:
$\lim _{x \rightarrow \mathrm{a}} \mathrm{f}(x)=\mathrm{f}(\mathrm{a})$ or $\lim _{x \rightarrow \mathrm{a}} \mathrm{f}(\mathrm{a}-\mathrm{h})=\mathrm{f}(\mathrm{a})$.
ii) If one of the previous conditions of continuity is not satisfied at a point the function is said to be discontinuous at that point.
Example: let ( $f$ ) be defined by
$f(x)= \begin{cases}\frac{x^{2}-4}{x-2} & ; x \neq 2 \\ 5 & ; x=2\end{cases}$
Then $(f)$ is discontinuous at $\boldsymbol{x}=\mathbf{2}$ since
i) $\quad f(2)=5$ defined
ii) $\lim _{x \rightarrow 2} f(x)=\lim _{x \rightarrow 2} \frac{x^{2}-4}{x-2}$

$$
\begin{aligned}
& \quad=\lim _{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)} \\
& =4 \text { exists. }
\end{aligned}
$$

iii) But $\lim _{x \rightarrow 2} f(x) \neq f(2)$

Since $4 \neq 5$
$\therefore(f)$ is discontinuous at $\boldsymbol{x}=\mathbf{2}$.

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### 7.3.Derivative of function :

Derivative of a function is one of the central concepts of calculus which has important applications in different fields such as mechanics, physics, economics, and statistics. the idea of dervative of a function ( $\boldsymbol{f}$ ) came from the idea of the gradient (slope ) of the tangent line to the graph of the function specified by the equation $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ through the idea of the rate of change of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$.

Different approaches of introducing the concept of derivative of a function which lead to the formal definition of this concept are followed by different school mathematics text books, the most convenient approach to the secondary school pupils is found to be the gradient ( slope ) of secant line approach through an experimental method involves drawing and measuring and calculating the gradients of a secant line to the graph of a given function and relating that to the gradiant of a tangent line to the curve as shown in the figure (7.11):


Fig.(7.11)
from this figure it can be seen that if $x$ change by a small amount $\Delta \mathbf{x}$ (called daltax ), $\boldsymbol{y}$ changes by a small amount $\Delta \boldsymbol{y}$, then the quotient $\frac{\Delta \boldsymbol{y}}{\Delta \boldsymbol{x}}=\frac{\text { change in } y}{\text { change in } x}$
is called the average rate of change of $(\boldsymbol{y})$ with respect to $(\boldsymbol{x})$, it can be seen that $\frac{\Delta y}{\Delta x}$ equals the gradient ( slope ) of the secant line $\mathbf{P Q}$, it can also be seen that as the point $\mathbf{Q}$ moves a long the curve towards the point $\mathbf{P}$ the secant line $\mathbf{P Q}$ rotates (revolves) a bout the point $\mathbf{P}$ toward its limiting position, the tangent line to the curve at $\mathbf{P}$, this can be represented algedraically as follows:

$$
\begin{aligned}
& y=f(x) \longrightarrow y+\Delta y=f(x+\Delta x) \\
& \therefore \Delta y=f(x+\Delta x)-y \\
& \quad=f(x+\Delta x)-f(x) \\
& \therefore \frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}
\end{aligned}
$$

This is called the rate of change of ( $\mathbf{y}$ ) with respect to ( $\mathbf{x}$ ) which equals the slope of the secant line to the curve, and

$$
\lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}=\lim _{\Delta x \rightarrow 0} \frac{\mathrm{f}(\mathrm{x}+\Delta x)-f(x)}{\Delta \mathrm{x}}
$$

Is called the instantaneous rate of change of $(\mathbf{y})$ with respect to $(\mathbf{x})$ which equales the gradient ( slope ) of the tangent line to the curve $\boldsymbol{y}=f(\boldsymbol{x})$.

### 7.3.1.The formal definition of the derivative :

The concept of dervative of a function is then defined using the idea of limits and slope of tangent to the graph of $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ as follows :
The dervative of the function $(\boldsymbol{f})$ specified by an equation $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ with respect to $x$ is

$$
f^{\prime}(x)=\lim _{\Delta x \rightarrow 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}
$$

## Notes:

1) If $\boldsymbol{x}_{1}$, and $\boldsymbol{x}_{2}$ are two real numbers the difference $\left(\boldsymbol{x}_{2}-\boldsymbol{x}_{1}\right)$ is denoted by $(\Delta \boldsymbol{x}, \operatorname{read}$ as delta $\boldsymbol{x})$ and called an increment or decrement according to the signe.
2) The derivative of the function specified by an equation $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ may be represented by any one of the following notation (symbols) $f^{\prime}, f^{\prime \prime}$ $, f^{\prime}(x), y^{\prime}, \frac{d y}{d x}, \frac{d}{d x} f(x), D_{x} y, D_{x} f(x)$.
3) The notation $f^{\prime}$ was the first used by newton, then the notation $\frac{\Delta y}{\Delta x}$, $\frac{\mathrm{d} y}{\mathrm{~d} x}$, was used by leibniz later on, the notation $f^{\prime}(x)$ was introduced more than ( 100 years) after newton and leibniz, and is read as the dervative of $f$ with respect to $(\boldsymbol{x})$ or ( $\boldsymbol{f}$ prime of $\boldsymbol{x}$ ).
4) The symbol $\frac{\mathrm{d} y}{\mathrm{~d} x}$ should be regarded as a single entity and not as quotient (fraction) of two quantities $\mathbf{d} \boldsymbol{y}$ and $\mathbf{d x}$.
5) The word derivative implies that $\left(f^{\prime}\right)$ is derived from $(f)$
6) A function is said to be differmatiable at $\boldsymbol{x}=\boldsymbol{a}$ if it has a derivative there that is $\lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ exists at that point.
7) If a function ( $\boldsymbol{f}$ ) is differentiable at a point, it is continuous at that point.
8) The gradient ( slope ) of the curve representing the graph of the function specified by the equation $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ at a point $\mathbf{P}$ is fined to be the gradient (slope) of the tangent line to this curve at that point which is found to be $\quad f^{\prime}(x)$ i.e. equals
$f^{\prime}(x)=\lim _{\Delta x \rightarrow 0} \frac{\mathrm{f}(x+\Delta x)-\mathrm{f}(x)}{\Delta x}$
9) There is an alternative form of the derivative fornula obtained by replacing $\Delta \mathbf{x}$ by $h$ to give $f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ which is easier to handle and use than the original formula.
10) If the differentiation process is applied successively ( n ) times to a function and its dervatives we produce each time a new derivative, the new derivatives are called higher derivatives and dinoted by the following nofations :
i) $\quad f^{\prime}(x), y^{\prime}, \frac{d y}{d x}, D_{x} y$, is called first derivative
ii) $\quad f^{\prime \prime}(x), y^{\prime \prime}, \frac{\boldsymbol{d}^{2} \boldsymbol{y}}{\boldsymbol{d x ^ { 2 }}},, D_{x}^{2} \boldsymbol{y}$ is called the second derivative
iii) $\quad f^{\prime \prime \prime}(x), y^{\prime \prime \prime}, \frac{d^{3} y}{d x^{3}}, D^{3}{ }_{x} y$ is called the third derivative
iv) $f^{n}(x), y^{\boldsymbol{n}}, \frac{\boldsymbol{d}^{n} \boldsymbol{y}}{\boldsymbol{d} x^{n}}, D^{n}{ }_{x} y$ is called the nth derivative
v) The process of finding the dervative of a function is called differentiation the symbols $\boldsymbol{D}_{\boldsymbol{x}}, \frac{\boldsymbol{d}}{\boldsymbol{d} \boldsymbol{x}}$ are called differentiation operators, the
branch of mathematics which deals with the calcultion of the derivatives of functions is called differential calculus.

### 7.3.2.Differentiation formulas :

The basic method of finding the derivate of a function is to apply the definition of a derivative $f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+\boldsymbol{h})-f(x)}{\boldsymbol{h}}$

This definition was used to develop formulas which can be used as an easy rules for finding derivatives for various types of functions much more rapidly than applying the definition, we will list these formulas ( rules ) here as follows:
a) Differentiation formulas for algebraic functions: if $\boldsymbol{f}$ and $\mathbf{g}$ are two differentiable functions of $\boldsymbol{x}$ and $\mathbf{c}$ is a real number then ;

1) $\left[\mathrm{c}(f(x)]^{\prime}=\mathrm{c} f^{\prime}(x)\right.$.

This formula states that the derivative of a constant times a function equals the constant times the dervative of this function
2) $[\mathbf{c}]^{\prime}=\mathbf{0}$

This formula states that the dervative of aconstant equals zero.
3) $\left[x^{n}\right]^{\prime}=n x^{n-1}$

This is a basic formula of differential calulus which is valid when $n$ is replaced by any non zero rational number
4) $[f(x)+g(x)]^{\prime}=f^{\prime}(x)+g^{\prime}(x)$

This formula states that the derivative of the sum of two function equals the sum of the derivatives of these two functions.
5) $[f(x)-g(x)]^{\prime}=f^{\prime}(x)-g^{\prime}(x)$

This formula states that the dervative of the difference of two functions equals the difference of the derivatives of the these two functions.
6) $[f(x) \cdot g(x)]^{\prime}=f(x) \cdot g^{\prime}(x)+g(x) \cdot f^{\prime}(x)$

This formula states that the derivative of the product of two functions equals the first time the derivative of the second plus the second times the derivative of the first.
7) $\left[\frac{f(x)}{g(x)}\right]^{\prime}=\frac{g(x) \cdot f^{\prime}(x)-f(x) \cdot g^{\prime}(x)}{[g(x)]^{2}}$

This formula states that the derivative of the quotient of two functions equals the denominator times the derivative of the numentor minus the
numerator times the derivative of the denominator, all divided by the squake of the denominutor.
8) If $\boldsymbol{u}=\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{y}=\mathbf{g}(\boldsymbol{u})$ are differentiable functions then the derivative of the composite function $\mathrm{g}[f]=($ gof $)$ defined by $\boldsymbol{y}=\mathrm{g}[f(\boldsymbol{x})]$ can be obtained by the chain rule using leibniz notation $\frac{d y}{d x}=\frac{d y}{d u} \cdot \frac{d u}{d x}$.
9) Using the chain rule we can derive an easy rule called the power formula for differentiation which is used in finding the derivative of a power of a function of $\boldsymbol{x}$ :
10)

$$
\begin{aligned}
& {\left[f(x)^{n}\right]^{\prime}=n\left(f(x)^{n-1}\right) \cdot f^{\prime}(x) \text { or }} \\
& \left(u^{n}\right)^{\prime}=n u^{n-1} \cdot u^{\prime}, \text { where } u=f(x)
\end{aligned}
$$

This short formula states that the derivative of a function( inside a bractice) of a power of n is equal to the derivative of the bractice times the derivative of what is inside the bractice.
11) Using the short formula ( rule ) derived from the chain rule we can derive a special formula ( rule ) for the derivative of the nth root of a function to the power of m as :

$$
\begin{aligned}
& {\left[\sqrt[n]{(\mathrm{f}(x))^{m}}\right]^{\prime}=\frac{\mathrm{m}}{\mathrm{n} \sqrt[n]{(\mathrm{f}(x))^{n-m}}} \cdot \mathrm{f}^{\prime}(x) \text { or }} \\
& {\left[\sqrt[n]{u^{m}}\right]^{\prime}=\frac{\mathrm{m}}{m^{n} \sqrt{u^{n-m}}} \cdot u^{\prime}}
\end{aligned}
$$

## Notes:

As special cases of this (rule) we can derive the following rules :
i) If $\mathbf{m}=\mathbf{1}$ we get the nth root rule :

$$
[\sqrt[n]{u}]^{\prime}=\frac{1}{n^{n} \sqrt[n]{n^{n}-1}} \cdot u^{\prime}
$$

ii) If $m=1$ and $n=2$ we get the square root rule

$$
[\sqrt{u}]^{\prime}=\frac{1}{2 \sqrt{u}} \cdot u^{\prime}
$$

## Examples :

Find the derivative of the following function by using the definition of derivative .
$\mathrm{f}(x)=3 x^{2}+1 \quad \mathrm{f}(x+\mathrm{h})=3(x+\mathrm{h})^{2}+1$
$\mathrm{f}^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$
$=\lim _{h \rightarrow 0} \frac{3(x+h)^{2}+1-\left(3 x^{2}+1\right)}{h}$
$=\lim _{h \rightarrow 0} \frac{3\left(x^{2}+2 x h+h^{2}\right)+1-3 x^{2}-1}{h}$
$=\lim _{h \rightarrow 0} \frac{3 x^{2}+6 x h+3 h^{2}+1-3 x^{2}-1}{h}$
$=\lim _{h \rightarrow 0} \frac{6 x h+3 h^{2}}{h}$
$=6 x$
$\mathrm{f}(x)=3 x^{2}+1 \Longleftrightarrow \mathrm{f}^{\prime}(x)=6 x$
find the derivative of each of the following functions by using an appropriate formula ( rule ):

1) $y=3 x^{4} \quad \Longrightarrow \mathrm{f}^{\prime}(x)=\frac{\mathrm{d} y}{\mathrm{~d} x}=12 x^{3}$
2) $y=2 \quad \Longrightarrow \mathrm{f}^{\prime}(x)=0$
3) $y=x^{3}-1 \Longleftrightarrow \frac{\mathrm{~d} y}{\mathrm{~d} x}=3 x^{2}$
4) $\mathrm{f}(x)=x^{2}+2 x^{3}+x \quad \Longrightarrow \mathrm{f}^{\prime}(x)=2 x^{2}+6 x^{2}+1$
5) $\mathrm{f}(x)=5 x^{3}-3 x^{2}-2 \longrightarrow \mathrm{f}^{\prime}(x)=15 x^{2}-6 x$
6) $y=\left(2+x^{3}\right)\left(x^{2}-1\right)$
$\frac{\mathrm{d} y}{\mathrm{~d} x}=\left(2+x^{3}\right)(2 x)+\left(x^{2}-1\right)\left(3 x^{2}\right)$
$=2 x\left(2+x^{3}\right)+3 x^{2}\left(x^{2}-1\right)$
$=4 x+2 x^{4}+3 x^{4}-3 x^{2}$
$=5 x^{4}-3 x^{2}+4 x$.
7) $y=\frac{2-x^{3}}{1+2 x^{2}}$

$$
\begin{aligned}
& \frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{\left(1+2 x^{2}\right)\left(-3 x^{2}\right)-\left(2-x^{3}\right)(4 x)}{\left(1+2 x^{2}\right)^{2}} \\
& \text { 8) } \mathrm{u}=3 x^{2}-1, y=\mathrm{u}^{3} \text { then } \\
& \frac{\mathrm{du}}{\mathrm{~d} x}=6 x \quad, \quad \frac{\mathrm{~d} y}{\mathrm{du}}=3 \mathrm{u}^{2} \\
& \therefore \frac{\mathrm{~d} y}{\mathrm{~d} x}=\frac{\mathrm{d} y}{\mathrm{du}} \cdot \frac{\mathrm{du}}{\mathrm{~d} x} \\
& =3 u^{2} .6 x \\
& =18 x . u^{2} \\
& =18 x\left(3 x^{2}-1\right)^{2} \\
& \text { 9) } y=\left(x^{2}-1\right)^{4} \\
& \frac{\mathrm{~d} y}{\mathrm{~d} x}=4\left(x^{2}-1\right)^{3} .2 x \\
& =8 x\left(x^{2}-1\right)^{3}
\end{aligned}
$$

10) i) $y=\sqrt[5]{\left(2-x^{2}\right)^{2}} \Rightarrow \mathrm{n}=5, \mathrm{~m}=2$

$$
\therefore \frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{2}{5 \sqrt[5]{(2-x 2)^{3}}} \cdot(-2 x)=\frac{-4 x}{5 \sqrt[5]{(2-x 2)^{3}}}
$$

ii) $y=\sqrt[3]{\left(x^{2}-3\right)^{2}}$

$$
\begin{aligned}
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\frac{1}{3 \sqrt[3]{\left(x^{2}-3\right)^{2}}} \cdot 2 x \\
& =\frac{2 x}{3 \sqrt[3]{\left(x^{2}-3\right)^{2}}}
\end{aligned}
$$

iii) $y=\sqrt{x^{3}-1}$

$$
\begin{aligned}
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\frac{1}{2 \sqrt{x^{3}-1}} \cdot 3 x^{2} \\
& =\frac{3 x 2}{2 \sqrt{\left(x^{3}-1\right)}}
\end{aligned}
$$

b) Differentiation formula for trigonometric functions :

The derivatives of the trigonometric functions are derived from that of the sin function by applying the definition of derivative to the equation $\boldsymbol{y}=\boldsymbol{\operatorname { s i n }} \boldsymbol{x}$, they are as shown in the table(7.1).

Table(7.1)

|  | $f(x)$ | $f^{\prime}(x)$ |
| :--- | :---: | :---: |
| 1 | $\sin x$ | $\cos x$ |
| 2 | $\cos x$ | $-\sin x$ |
| 3 | $\tan x$ | $\sec ^{2} x$ |
| 4 | $\cot x$ | $-\operatorname{cosec} x$ |
| 5 | $\sec x$ | $\sec x \cdot \tan x$ |
| 6 | $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cdot \cot x$ |

## Notes:

i) The derivative rules for the algebraic functions are used here.
ii) Put $\boldsymbol{\operatorname { s i n }}^{\boldsymbol{n}} \boldsymbol{x}=(\boldsymbol{\operatorname { s i n }} \boldsymbol{x})^{\boldsymbol{n}}$ to make it easy to use the power formula. Example:
Find $\frac{\mathrm{d} y}{\mathrm{~d} x}$ for the function

$$
\begin{aligned}
& y=\sin ^{3} x^{2} \\
& \text { Write } y=\left(\sin x^{2}\right)^{3} \\
& \therefore \quad \frac{\mathrm{~d} y}{\mathrm{~d} x}=3\left(\sin x^{2}\right)^{2} \cdot \cos x^{2} \cdot 2 x \\
& \quad=6 x \sin ^{2} x^{2} \cdot \cos x^{2}
\end{aligned}
$$

c) Differentiation formulas for the inverse trigonometric functions :

As shown in the table (7.2) :

Table(7.1)

|  | $f(x)$ | $f^{\prime}(x)$ |
| :--- | :--- | :--- |
| 1 | $\arcsin x$ | $\frac{1}{\sqrt{1-x^{2}}}$ |
| 2 | $\arccos x$ | $-\frac{1}{\sqrt{1-x^{2}}}$ |
| 3 | $\arctan x$ | $\frac{1}{1+x^{2}}$ |
| 4 | $\operatorname{arccot} x$ | $-\frac{1}{1+x^{2}}$ |
| 5 | $\operatorname{arcsec} x$ | $\frac{1}{x \sqrt{x^{2}-1}}$ |
| 6 | $\arccos x$ | $-\frac{1}{x \sqrt{x^{2}-1}}$ |

## Example:

Find $\frac{\mathrm{d} y}{\mathrm{~d} x}$ for $y=\arcsin \left(2 x^{3}-1\right)$
$y=\arcsin \left(2 x^{3}-1\right)$
$\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{1}{\sqrt{1-\left(2 x^{3}-1\right)^{2}}} \cdot 6 x$
d) Differentiation formulas for the exponential and logarithmic functions as shown in the table(7.3) :

Table(7.1)

|  | $f(x)$ | $f^{\prime}(x)$ |
| :---: | :---: | :---: |
| 1 | $\log _{a} x$ | $\frac{1}{x} \log _{a} e=\frac{1}{x \ln a}$ |
| 2 | $\operatorname{Ln} x$ | $\frac{1}{x}$ |
| 3 | $a^{x}$ | $a^{x} \ln a$ |
| 4 | $e^{x}$ | $e^{x}$ |

Notes: the following properties of logarithms can be used to simplify the function before applying the appropriate formula.
i) $\quad \log _{\mathrm{a}} x^{\mathrm{n}}=\mathrm{n} \log _{\mathrm{a}} x$
ii) $\quad \log _{\mathrm{a}}\left(x_{1}, x_{2}\right)=\log _{\mathrm{a}} x_{1}+\log _{\mathrm{a}} x_{2}$
iii) $\log _{a}\left(\frac{x_{1}}{x_{2}}\right)=\log _{a} x_{1}-\log _{a} x_{2}$

## Example :

Find $\frac{\mathrm{d} y}{\mathrm{~d} x}$ for each of the following

1) $y=\log _{10}\left(2 x^{3}-1\right)$
$\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{1}{\left(2 x^{3}-1\right) \operatorname{Ln} 10} \cdot 6 x$
2) $y=\operatorname{Ln}\left(x^{3}-2 x^{2}+1\right)^{4}$
$y=4 \operatorname{Ln}\left(x^{3}-2 x^{2}+1\right)$
$\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{4.3}{\left(x^{3}-2 x^{2}+1\right)} \cdot\left(3 \mathrm{x}^{2}-4 \mathrm{x}\right)=3 \frac{\left(3 x^{2}-4 x\right)}{\left(x^{3}-2 x^{2}+1\right)} \cdot 4$
3) $y=a^{2 x^{3}}$
$\frac{\mathrm{d} y}{\mathrm{~d} x}=a^{2 x^{3}} \ln a .6 x^{2}=6 x^{2} . a^{2 x^{3}} \ln a$
4) $y=e^{x^{3}}$

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=e^{x^{3}} \cdot 3 x^{2}=3 x^{2} e^{x^{3}}
$$

### 7.4. Integral of function:

The integral calculus is the second branch of calculus which is dealing with the process of finding the integral of a function this operation is then called integration. to integrate means to sum that is to find a value by the addition of parts or elements. text books of school mathematics for pupils under sixten years of age varied considerably in their treatement of integration some donot consider it at all some started with antidifferentiation, others emphasized the idea of area as a step to define integration, others introduced integration as the reverse operation of differentiation and later on they used integration as a tool to calculate areas.
From the a bove argument it appears that there are two approaches of introducing integration one of them is geometrical used the idea of area, the other is algebrical used the idea of antidifferentiation each of these approaches leeds to a specific type of integration which leeds to a specific definition to this concept.

### 7.4.1.The definite integral ( Riemann integral ):

The definite integral of a function $(\boldsymbol{f})$ specified by $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ is the area under the graph of $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$. see the figure (7.12).


Fig.(7.12)

As shown in the figure the region under the graph of $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ is divided in to a number of strips eash in the shap of a rectangle of width equals ( $\Delta x$ ).

Then the area under the graph of $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$ is

$$
\begin{aligned}
\mathrm{A} & =\lim _{\Delta x \rightarrow 0} \sum_{a}^{b} y \Delta x \\
& =\lim _{\Delta x \rightarrow 0} \sum_{a}^{b} f(x) \Delta x
\end{aligned}
$$

Then this summation is represented and denoted by $\int_{a}^{b} y d x=\int_{a}^{b} f(x) d x$

Thus the definite integral of the function $(f), \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{d} \boldsymbol{x}$ is the shaded area shown in the previous figure.

## Notes:

1-The symbol ( $\sum$ ) is the Greek caplital letter ( sigma) which is frequently used to denote the sum of several terms, introduced by Euler since ( 1755).
2- The symbol ( $\int$ ) is called an integral sign it is a stylization of the letter ( $\mathbf{S}$ ) that was used in Leibniz's time to indicate a summation.
3- The symbol $\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{d} \boldsymbol{x}$ is read as the definite integral of $\boldsymbol{f}(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ from $\boldsymbol{x}=\mathbf{a}$ to $\boldsymbol{x}=\mathbf{b}$.
4- The expression $\boldsymbol{f}(\boldsymbol{x})$ is called the integral, $\boldsymbol{x}$ the variable of integration, $\mathbf{a}$ and $\mathbf{b}$ are called respectivelly the lower and upper limits or boundries of integration.
5- The definite integral $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}(\boldsymbol{x}) \mathbf{d} \boldsymbol{x}$ is called Riemann definite integral due to the German mathematician (Bernhard Riemann 1826-1866).
6- If the $\lim _{\Delta x \rightarrow 0} \sum_{\mathbf{a}}^{\mathbf{b}} f(x) \Delta x=\int_{a}^{b} f(x) d x$
Exists the function $(\boldsymbol{f})$ is said to be Riemann integrable on the interual [a,b].

### 7.4.2.Properties of definite integral:

If $f$ and $g$ are continuous function on the interval of integratio $[a, b]$ then the following properties hold:

1) $\int_{a}^{a} f(x) d x=0$
2) $\int_{a}^{b} f(x) d x=-\int_{b}^{a} f(x) d x$
3) $\int_{a}^{b} c f(x) d x=c \int_{a}^{b} f(x) d x, c$ is a constant.
4) $\int_{a}^{b}[f(x) \pm g(x)] d x=\int_{a}^{b} f(x) d x \pm \int_{a}^{b} g(x) d x$
5) 

$$
\int_{a}^{c} f(x) d x+\int_{c}^{b} f(x) d x=\int_{a}^{b} f(x) d x
$$

The definite integral $\int_{a}^{b} f(x) d x$ is by definition the area bounded by the graph of $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$, the $\boldsymbol{X}$ - axis, and the lines $\boldsymbol{x}=\mathbf{a}, \boldsymbol{x}=\mathbf{b}$, $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x}) d \boldsymbol{x}$ is positive indicates that the area lies above the $\boldsymbol{X}$ - axis, if it is negative the area lies below the $X$-axis; (see the figure (7.13-a)).
6) The definite integral $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{g}(\mathbf{y}) \mathbf{d y}$ is by definition the area bouneded by the graph of $\boldsymbol{x}=\mathbf{g}(\boldsymbol{y})$, the $\boldsymbol{Y}$ - axis, the lines $\boldsymbol{y}=\mathbf{a}, \boldsymbol{y}=\mathbf{b}$.
$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{g}(\boldsymbol{y}) \mathbf{d} \boldsymbol{y}$ is positive indicates that the area lies to the right of the $\boldsymbol{Y}-$ axis, if it is negative the area lies to the left of the $\boldsymbol{Y}$ - axis, (see the figure (7.13-b)).


Fig.( 7.13-a )


Fig. (7.13-a )

### 7.4.3.The indefinite integral ( anti-derivative ) :

The approache used the idea of anti-differentiation looks to differentiation and inttegration as inverse processes, that is if we start with a continuous function $(\boldsymbol{f})$ and integrate to obtain $\int_{\boldsymbol{c}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{d} \boldsymbol{x}$ and then we differentiate the resulting function we obtain the original function ( $f$ ), thus the differentiation has nullified the integration. On the other hand if we start with the function ( $\boldsymbol{f}$ ) and differentiate it to obtain $\boldsymbol{f}^{\prime}(\boldsymbol{x})$ and then integrate the resulting function we obtain the original function ( $\boldsymbol{f}$ )thus intrgration nullified the differentiation, as shown in the following diagrams (figure 7.14).


Fig.( 7.14 )

## Example:




## Example :

$$
x^{3} \longrightarrow 3 x^{2} \longrightarrow \frac{3 x^{2+1}}{2+1}=\frac{3 x^{3}}{3}=x^{3}
$$

The fundamental theorem of calculus and the previous argument about the relationship between the processes of differentiation and integration indicate that to evaluate an integral we must obtain an anti-derivative formula from differentiation formula by reversing the operation, such that $\boldsymbol{D}_{\boldsymbol{x}}\left(\boldsymbol{x}^{\mathrm{n}}\right)=\mathbf{n} \boldsymbol{x}^{\mathrm{n}-1}$ is a differentiation formula reverse gives the formula
$D_{x}^{-1}\left(x^{n}\right)=\frac{x^{n+1}}{n+1}+\mathrm{c}, \mathrm{n} \neq-1$
Example: $D_{x}^{-1}\left(x^{4}\right)=\frac{x^{4+1}}{4+1}+\mathrm{c}=\frac{x^{5}}{5}+\mathrm{c}$
According to that the fundamental theoren of calculus states that:

$$
\begin{aligned}
\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(x) \mathrm{d} x & \left.=D_{x}^{-1}[f(x)]_{a}^{b}=\mathrm{G}(x)\right]_{a}^{b} \\
& =\mathrm{G}(\mathrm{~b})-\mathrm{G}(\mathrm{a})
\end{aligned}
$$

Example $: \int_{2}^{4} x^{2} \mathrm{~d} x=\left.D_{x}^{-1}\left(x^{2}\right)\right|_{2} ^{4}=\left.\frac{x^{3}}{3}\right|_{2} ^{4}$

$$
=\frac{(4)^{3}}{3}-\frac{(2)^{3}}{3}=\frac{64}{3}=\frac{8}{3}=\frac{56}{3}
$$

The above ideas suggest the following definitions :

1) Anti -derivative of a function :

The anti-derivative of a function $(\boldsymbol{f})$ on an interval [a, b ] is a function (G) whose derivative $G^{\prime}(x)=f(x)$ for all $\boldsymbol{x} \in[a, b]$.
2) The indefinite integral of a function (f) on an interval [a, b] is an anti-derivative of $(\boldsymbol{f})$ on [ a , b ], it is also called the general indifinite integral of $(\boldsymbol{f})$ and is denoted by $\int f(x) d x$
Notes :

1) The anti-derivative of a function is called an indefinite integral.
2) Every continuous function ( $\boldsymbol{f}$ ) has infinitley many anti-dervatives that is many indefinite integrals, so that if $(G)$ is an anti-derivative of $(\boldsymbol{f})$ on the interval [ $\mathrm{a}, \mathrm{b}$ ] then: $(\mathrm{G}+\mathrm{c})$ is also an anti-derivative of (f) where c is a constant and is written as $\int f(x) \mathrm{d} x=\mathrm{G}(x)+\mathrm{c}$.
3) $\int(f) \mathrm{d} x=\mathrm{G}(x)+\mathrm{c}$
 $D_{x}[\mathrm{G}(\mathrm{x})+\mathrm{c}]=f(x)$

This means that if we differentiate the anti-derivative we will get the original function $(\boldsymbol{f})$
4) $\int f(x) d x=f(x)$

$$
D_{x}[f(x)]=f^{\prime}(x)
$$

This means that if we integrate the dervative of $(\boldsymbol{f})$ we will get the originale function $(\boldsymbol{f})$
5) $G^{\prime}(x)=f(x)$ means that $(\mathbf{G})$ is an anti-derivative of $(\boldsymbol{f})$ which is also called a primitive. the word primitive suggests that $(\boldsymbol{f})$ is derived from (G) and the word anti-derivative implies that $(\mathbf{G})$ is obtained from $(\boldsymbol{f})$ by reversing the process of differentiation.
6) We get an anti-derivative formula by reversing the differentiation formula associated with it for example :
$D_{x}\left(\frac{x^{n+1}}{n+1}\right)=x^{n} ; \mathrm{n} \neq-1$ means that $\frac{x^{n+1}}{n+1}$ is a primitive or an antiderivative of $\boldsymbol{x}^{\boldsymbol{n}}$ then;
$D_{x}^{-1}\left(x^{n}\right)=\frac{x^{n+1}}{n+1}+c, \mathrm{n} \neq-1$ which is written in Leibniz notation as $\int x^{n} \mathrm{~d} x=\frac{x^{n+1}}{n+1}+c ; \mathrm{n} \neq-1$ which is called the general anti-derivative of $\boldsymbol{x}^{n}$ or the definite integral of $\boldsymbol{x}^{\boldsymbol{n}}$, this formula is refered to as the power formula for intigration.

## Example:

$$
D_{x}^{-1}\left(x^{2}\right)=\int x^{2} \mathrm{dx}=\frac{x^{2}}{3}+\mathrm{c}
$$

7)The symbole $\mathrm{D} x$ is a differentiation operator while the symbol $\boldsymbol{D}_{x}^{-\mathbf{1}}$ is an inverse opertor and indicates that the operation which is inverse to differentiation is to be performed on the expression which it is operating .

Example : $D_{x}^{-1}\left(x^{3}\right)=\frac{x^{4}}{4}+\mathrm{c}$
8)When we have found a formula for $\boldsymbol{G}(\boldsymbol{x})$ interms of $(\boldsymbol{x})$ for which:

$$
\int f(x) d x=G(x)+c
$$

We say that we have evaluated $\int \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{d} \boldsymbol{x}$, or we have found the integral of $\boldsymbol{f}(\boldsymbol{x})$.

### 7.4.4.Table of commonly used integration formulas :

We will present some elementary rules that will help us to evaluate indefinite inlegrals by finding anti-derivative. we note that the following properties hold.

1) $\int c f(x) d x=\mathrm{c} \int f(x) d x$; c is a real number.
2) $\quad \int[f(x) \pm \mathrm{g}(x)] \mathrm{dx}=\int f(x) \mathrm{d} x \pm \int \mathrm{g}(x) \mathrm{d} x$
3) $\quad \int u d v=u v-\int v d u \quad$ where $\mathbf{u}$ and $\mathbf{v}$ are functions of $\boldsymbol{x}$, this formula is called integration by parts formula which is concerned with the integration of the product of two functions one is $\mathbf{u}$ and the other is $\mathbf{d v}$.

Commonly used formulas
$\int \operatorname{Ln} u d u=u \ln u-u+c$

$$
\int e^{u} \mathrm{du}=e^{u}+\mathrm{c}
$$

$$
\begin{aligned}
& \int \sin ^{2} u d u=\frac{u}{2}+\frac{\sin 2 \mathrm{u}}{4}+\mathrm{c} \\
& \int \cos ^{2} u d u=\frac{u}{2}-\frac{\sin 2 \mathrm{u}}{4}+\mathrm{c} \\
& \int \frac{\mathrm{du}}{\mathrm{u}^{2}-\mathrm{a}^{2}}=\frac{1}{2 \mathrm{a}} \ln \left|\frac{\mathrm{u}-\mathrm{a}}{\mathrm{u}+\mathrm{a}}\right|+\mathrm{c} \\
& \int \frac{\mathrm{du}}{\mathrm{a}^{2}-\mathrm{u}^{2}}=\frac{1}{2 \mathrm{a}} \ln \left|\frac{\mathrm{u}+\mathrm{a}}{\mathrm{u}-\mathrm{a}}\right|+\mathrm{c} \\
& \int \frac{d u}{\sqrt{\mathrm{u}^{2}+\mathrm{a}^{2}}}=\ln \left|\mathrm{u}+\sqrt{\mathrm{a}^{2}+\mathrm{u}^{2}}\right|+\mathrm{c} \\
& \int u^{n} d u=\frac{u^{n+1}}{n+1}+\mathrm{c}, \mathrm{n} \neq-1 \\
& \int \frac{\mathrm{du}}{\mathrm{u}}=\ln |\mathrm{u}|+\mathrm{c} \\
& \int \mathrm{udv}=\mathrm{uv}-\int \mathrm{v} d u
\end{aligned}
$$

$$
\int \sin u d u=-\cos u+c
$$

$$
\int \cos u d u=-\sin u+c
$$

$$
\int \tan u d u=\ln |\cos u|+c
$$

$$
\int \cot u d u=\ln |\sin u|+c
$$

$$
\int \sec u d u=\ln |\sec u+\tan u|+c
$$

$$
\int \csc u d u=\ln |\csc u-\cot u|+c
$$

$$
\int \sec ^{2} u=\tan u+c
$$

$$
\int \operatorname{cosec}^{2} u d u=-\cot u+c
$$

$$
\int \sec u \tan u d u=\sec u+c
$$

$$
\int \csc u \cot u d u=-\csc u+c
$$

$$
\int \frac{d \mathrm{u}}{a^{2}-\mathrm{u}^{2}}=\frac{1}{\mathrm{a}} \tan ^{-1} \frac{\mathrm{u}}{\mathrm{a}}+\mathrm{c}
$$

$$
\int \frac{\mathrm{du}}{\sqrt{a^{2}-u^{2}}}=\sin ^{-1} \frac{\mathrm{u}}{\mathrm{a}}+\mathrm{c}
$$

$$
\int \sqrt{u^{2} \pm a^{2}} \quad d u=\frac{u}{2} \sqrt{u^{2} \pm a^{2}} \pm \frac{a^{2}}{2} \ln \left|u+\sqrt{u^{2} \pm a^{2}}\right|+c
$$

$$
\int \sqrt{a^{2}-u^{2}} \quad d u=\frac{u}{2} \sqrt{a^{2}-u^{2}}+\frac{a^{2}}{2} \sin ^{-1} \frac{u}{a}+c
$$

$$
\int \frac{\mathrm{du}}{\sqrt{\mathrm{u}^{2}-\mathrm{a}^{2}}}=\ln \left|\mathrm{u}+\sqrt{\mathrm{u}^{2}-\mathrm{a}^{2}}\right|+\mathrm{c}
$$

## Unit.8.The graphical method of solving equations and inequalities.

8.1 - The graphical method of solving equalities.
8.1.1-Linear equations in two variables.
8.1.2-System of linear equations in two variables.
8.1.3- Linear equations in one variable.
8.1.4- Quadratic equations in one variable.
8.2 - The graphical method of solving inequalities.
8.2.1-Linear inequalities in one variable.
8.2.2- Quadratic inequalities in one variable.
8.2.3- Linear inequalities in two variables.

## Unit . 8 The graphical method of solving Equations and in equalities

## Introduction

We indicated in the algebraic unit. 3 that the equation is an open sentence having two equal sides containing numbers and variables connected with multiplication and addition operations, similarly the inequality is an open sentence having two in equal (not equal) sides, the solutions of an equation or an inequality are the values of the variables which make the sentence a true statement, (satisfy it).

The solution set is the set of all solutions of the equation or inequality.
The graph of an equation or an inequality is the graph of its solution set which is the set of all points whose co-ordinates satisfy that equation or inequality, this shows that the equation is the algebraic analog of the graph and the graph is the geometric analog of the equation ,this indicates that the algebraic method of solving equations and inequalities shows the solutions in an abstract and formal form (algebraic) while the graphical method shows the solutions in a simi concrete form (geometric) it is the best way of seeing the relationship between the variables geometrically.

Graphs are used to convey in a simple pictorial and immediate way ideas which otherwise would require many words, figures or symbols to describe. this is due to that we can frequently visualize a complete graph though we have only enough space to draw a part of it recognize that the graphical representation is one of the most valuable mathematical tools used by scientists, economists, governments, industry, and commercialists, we will add here that for the citizen who cannot read a graph is a handicapped member of society. Thus, the growth of graphical literacy, both in the reading of graphs and in the interpretation of their messages, should be one of the main objectives of graphical work for all pupils, this is due to the fact that several graphs are used so often in mathematics and science that they have given names, among such graphs are the parabola, hyperbola, ellipse, logarithmic curve, and many more.

In this unit we will discuss the graphical method of solving equations and inequalities

### 8.1.The graphical method of solving equations:

### 8.1.1 Linear equations in two variables

An equation in the form $\boldsymbol{a x}+\boldsymbol{b y}+\boldsymbol{c}=\mathbf{0}, \boldsymbol{a}, \boldsymbol{b} \neq \mathbf{0}$ is called a linear (first degree) equation in two variables $\boldsymbol{x}$ and $\boldsymbol{y}$. the solution set of such an equation is the set of all solutions of this equation. which is in the form of a set of ordered pairs of numbers $(\boldsymbol{x}, \boldsymbol{y})$ which make the equation true i.e. satisfy the equation. The graph of the equation is the graph of its solution set, i.e. the graph of an equation in the form $\boldsymbol{a x}+\boldsymbol{b} \boldsymbol{y}+\boldsymbol{c}=\mathbf{0}$ is the graph of the set $\boldsymbol{S}=\{(\boldsymbol{x}, \boldsymbol{y}): \boldsymbol{a x}+\boldsymbol{b} \boldsymbol{y}+\boldsymbol{c}=\mathbf{0}\}$. To graph the set $(\boldsymbol{S})$ we have to find at least two ordered pairs (two points) in this set by giving the variables $\boldsymbol{x}$ or $\mathbf{y}$ some selected values and solving for $\boldsymbol{x}$ or $\boldsymbol{y}$ in each case. Then these ordered pairs are graphed, notice that connecting these points gives straight line. Thus, the graph of this type of equations is a straight line and each point on it forms an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ which is a solution to this equation. Thus, the graphical solution set of this equation is the set of all Points on this straight line which is an infinite set, being the line extends to infinity from both sides.

## Example:

Discuss the graph of the following Linear equation $\boldsymbol{x}-2 \boldsymbol{y}+4=0$.
Solution: The graph of this equation is the graph of the set

$$
S=\{(x, y): x-2 y+4=0\} .
$$

Find two ordered pairs (points) on this line, it is more convent to find the $\boldsymbol{x}$ and $\boldsymbol{y}$ intercepts as shown in the following table:


Notice that in case of a straight line two point are enough. Then the above points are graphical in the figure (8.1).

The solution set graphically is the set of all points on the line (L), see the figure (8.1).


Fig.(8.1)

### 8.1.2.System of linear equations in two variables :

We indicated in the previous section that the graphical solution set of a linear equation in two variables in the form $\boldsymbol{a x}+\boldsymbol{b y}+\boldsymbol{c}=\mathbf{0}$ is all the points $(\boldsymbol{x}, \boldsymbol{y})$ on a straight line. there fore the solution set of a system of two linear equations $\boldsymbol{a}_{\mathbf{1}} \boldsymbol{x}+\boldsymbol{b}_{\mathbf{1}} \boldsymbol{y}+\boldsymbol{c c}_{\mathbf{1}}=\mathbf{0}, \boldsymbol{a}_{\mathbf{2}} \boldsymbol{x}+\boldsymbol{b}_{\mathbf{2}} \boldsymbol{y}+\boldsymbol{c}_{\mathbf{2}}=\mathbf{0}$, (which are called also a system of simultanieous linear equations ) is obtained graphically by graphing each equation of the system using the same co-ordinate axes for both on the same graph paper, we get two lines in the plane, then the solution of the system is represented graphically by the point of intersection of the lines representing each equation.

Note: Asystem of two linear equations in two variables may have one of the following possibilities:

1) One solution.

This is the case if the graphs of the equations are two intersecting lines (in one point) such a system is said to be independent.
2) No solution.

This is the case if the graphs of the equations are two parallel lines such a system is said to be inconsistent.
3) Infinitely many solutions.

This is the case if the graphs of the equations are one and the same line, such a system is said to be dependent.

## Examples :

Solve graphically each of the following systems of linear equations:

1) $\left\{\begin{array}{c}3 x+2 y-6=0 \\ x-y+4=0\end{array}\right.$

The solution set of this system is

$$
\{(x, y): 3 x+2 y-6=0\} \cap\{(x, y): x-y+4=0\}
$$

Graphing both equations using the same graph ( squared ) paper by finding two points on each ( $x$ and $y$ intercepts if possible ) as shown in the following tables and the figure (8.2).

| $x$ | 0 | 2 |
| :--- | :--- | :--- |
| $y$ | 3 | 0 |


| $x$ | 0 | -4 |
| ---: | :--- | ---: |
| $y$ | 4 | 0 |



Fig.(8.2)
$\therefore L_{1} \cap L_{2}$ is $P(-0.4,3.2)$
$\therefore$ The system is independent.
أ.د. أحمد العريفي الشارف
2) $\left\{\begin{aligned} 3 x+2 y & =1 \\ 2 x-3 y & =-8\end{aligned}\right.$

Solution : Equations (1) and (2) are graphed by finding and then plotting two solutions of each equation as indicated in the figure (8.3), the point of intersection of these two lines appears to the point $(-1,2)$ Thus this system is independent.


Fig.(8.3)
3) $\left\{\begin{array}{l}2 x-y=7 \\ 4 x-2 y=3\end{array}\right.$

Solution : Equations (1) and (2) are graphed by finding and then ploting two solutions of each equation as indicated in the figure (8.4). Note that the two lines are parallel and therefore do not intersect. We conclude that the given system has no solution or alternatively, that its solution set is the empty set, $\varnothing$. Thus problem 3 is an example of an inconsistent system.


Fig.(8.4)
4) $3 x+2 y=5$
$9 x+6 y=15$

Solution : Equations (1) and (2) are graphed by finding and then plotting two solutions of each equation, as indicated in the figure (8.5). Note that the graphs of the two equations are one and the same line. We conclude that the given system has many solutions.
A system of linear equations such as the one in problem 4 is called a dependent system.


### 8.1.3. Linear equations in one variable.

To solve a linear equation in one variable ( first degree equation ) in the form $\boldsymbol{a x}+\boldsymbol{b}=\mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$ graphically we need another variable, so we will introduce a new variable called it $\boldsymbol{y}$ to obtain:

$$
y=a x+b=0, a \neq 0
$$

Then we divide the new form in to two parts, to obtain the following cases:
Case i): $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}=\mathbf{0}$, gives $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}$ and $\boldsymbol{y}=\mathbf{0}$
Find $x$ at the intersection of $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}$ and the $\boldsymbol{X}$-axis.
Case ii ): $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}=-\boldsymbol{b}$, gives $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}$ and $\boldsymbol{y}=-\boldsymbol{b}$
Find $\boldsymbol{x}$ at the intersection of $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}$ and $\boldsymbol{y}=-\boldsymbol{b}$
Case iii): $y=b=-a x$, gives $y=b$ and $y=-a x$
Find $x$ at the intersection of $\boldsymbol{y}=\boldsymbol{b}$ and $\boldsymbol{y}=-\boldsymbol{a} \boldsymbol{x}$
Case iv): $y=x+\frac{b}{a}=0$
Find $\boldsymbol{x}$ at the intersection of $\boldsymbol{y}=\boldsymbol{x}$ and $\boldsymbol{y}=-\frac{\boldsymbol{b}}{\boldsymbol{a}}$

$$
y=x=-\frac{b}{a}, \text { gives } y=x \text { and } y=-\frac{b}{a}
$$

## Notes:

1) Notice that each time we obtain equations of two straight lines thus the solution set can be obtained by graphing these two equations each time on the same graph paper, and the solution is the value of $\boldsymbol{x}$ at the point of intersection of the two lines.
2) Notice also that the solutions will be the same solution by all cases.
3) The last case iv $\boldsymbol{y}=\boldsymbol{x}$ and $\boldsymbol{y}=-\frac{\boldsymbol{b}}{\boldsymbol{a}}$ can be taken as a general case to solve graphically any equal in the farm $\boldsymbol{a x}+\boldsymbol{b}=\mathbf{0}$, we suggest that a standred form of the graph of $\boldsymbol{y}=\boldsymbol{x}$ (it is a line Harough ) is prepared and be avaliable to all pupiles all the time in this case the pupil may graph the line $\boldsymbol{y}=-\frac{\boldsymbol{b}}{\boldsymbol{a}}$ on that paper of $\boldsymbol{y}=\boldsymbol{x}$ and determine the value of $\boldsymbol{x}$ at their point of intersection.

## Example :

Solve graphically the equation $2 \boldsymbol{x}+\mathbf{5}=\mathbf{0}$.
Solution : we will discuss the graph of this equation using the four situations ( cases ); first put $\boldsymbol{y}=\mathbf{2 x}+\mathbf{5}=\mathbf{0}$ then use :
Case i): $y=2 x+5=0 \longrightarrow y=2 x+5$ and $y=0$
Find $x$ at the intersection of $\boldsymbol{y}=\mathbf{2 x}+\mathbf{5}$ and $\boldsymbol{X}$-axis

Figure (8.6-i) shows that the point of intersection is
$P_{1}(-2.5,0) \longrightarrow x=-\frac{5}{2}$ is the solution.
Case ii) $y=2 x=-5 \longrightarrow y=2 x$ and $y=-5$
Find $x$ at the intersection of $y=2 x$ and $y=-5$.
Fig (8.6-ii) shows that the point of intersection is $P_{2}(-2.5,5) \longrightarrow x=-2.5$ is the solution.
Case iii): $y=5=-2 x \longrightarrow y=5$ and $y=-2 x$.
Find $x$ at the intersection of $y=-2 x$ and $y=5$.
Fig (8.6-iii) shows that the point of intersection is
$P_{3}(-2.5,5) \longrightarrow x=-2.5$ is the solution.
Case iv): $y=x=-\frac{5}{2} \longrightarrow y=x$ and $y=-2.5$.
Find $x$ at the intersection of $y=x$ and $y=-\frac{5}{2}$
Figure (8.6-iv) shows that the point of intersection is $P_{4}(-2.5,-2.5) \longrightarrow x=-2.5$ is the solution.
Notice that the graphs of the four cases gave the same result which is the $x$ value at the point of intersection of the two lines is $x=-2.5$ which means that the solution of this equation is $x=-2.5$.
Notice also that the graphical method in this solution is more difficult than the algebraic method of solving this type of equations (isolatng method) shown in 3.3.1.



Fig.(8.6-iii)


Fig.(8.6-iv)

### 8.1.4.Quadratic equations in one variable :

To solve graphically a quadratic equation in one variable(second degree equation ) in the form $\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}=\mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$ we will apply the idea of graphing a quadratic function $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}, \boldsymbol{a} \neq \mathbf{0}$ in a similar way to that of solving the first degree equation ( linear equation ).

Disscussed in the last section (8.1.3), then we will introduce a new variable $\boldsymbol{y}$ to obtain $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}=\mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$. then we divide the new form in to two parts to obtain the following cases ( situations) :
Case i): $y=a x^{2}+b x+c=0 \longrightarrow y=a x^{2}+b x+c$ and $y=0$ Find $x$ at the intersection of $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}$ and $\boldsymbol{X}$ - axis
Case ii): $y=a x^{2}+b x=-c \longrightarrow y=a x^{2}+b x$ and $y=-c$
Find $x$ at the intersection of $\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b x}$ and $\boldsymbol{y}=-\boldsymbol{c}$
Case iii): $y=a x^{2}+c=-b x \longrightarrow y=a x^{2}+c$ and $y=-b x$
Find $x$ at the intersection of $y=a x^{2}+c$ and $y=-b x$
Case v): put the equation in the form :
$y=x^{2}+\frac{b}{a} x+\frac{c}{a}=0 \longrightarrow y=x^{2}=-\frac{b}{a} x-\frac{c}{a}=\frac{-b x-c}{a}$
Which give the simpler form
$\boldsymbol{y}=\boldsymbol{x}^{2}$ and $\boldsymbol{y}=\frac{-\boldsymbol{b} \boldsymbol{x}-\boldsymbol{c}}{\boldsymbol{a}}$

## Notes:

1) Each time we obtain an equation of a curve and a straight line, thus the solution is the value of $\boldsymbol{x}$ at the intersection of these two equations.
2) The solution will be the same by all cases.
3) The last case v can be taken as a general case in which the standared graph $\boldsymbol{y}=\boldsymbol{x}^{2}$ is prepared (it is a parabola in the standared position) and provided to pupil whos can graph the line $\boldsymbol{y}=\frac{-\boldsymbol{b} \boldsymbol{x}-\boldsymbol{c}}{\boldsymbol{a}}$ on the same graph paper of $\boldsymbol{y}=\boldsymbol{x}^{2}$.

## Example :

Solve graphically the equation $\boldsymbol{x}^{2}-2 \boldsymbol{x}-\mathbf{3}=\mathbf{0}$
Solution: we will discuss the graph of this equation using the four cases as follows.
Case i) put $y=x^{2}-2 x-3=0 \longrightarrow y=x^{2}-2 x-3$ and $y=0$ Find $\boldsymbol{x}$ at the intersection of these two graphs. as indicated in the figure (8.7).

| $\boldsymbol{x}$ | -2 | -1 | 0 | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\boldsymbol{y}$ | 5 | 0 | -3 | -4 | -3 | 0 |



Fig.(8.7)
$S=\{-1,3\}$
Case ii) put $y=x^{2}-2 x=3 \Longrightarrow y=x^{2}-2 x$ and $y=3$

Find $x$ at the intersection of these two graphs. as indicated in the figure (8.8).

| $\boldsymbol{x}$ | -2 | -1 | 0 | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\boldsymbol{y}$ | 8 | 3 | 0 | -1 | 0 | 3 |



Fig.(8.8)
$S=\{-1,3\}$

Case iii) put $y=x^{2}-3=2 x \Longrightarrow y=x^{2}-3$ and $y=2 x$
Find x at the intersection of these two graphs. as indicated in the figure

| $\boldsymbol{x}$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\boldsymbol{y}$ | 6 | 1 | -2 | -3 | -2 | 1 | 6 |

Case iv) put $y=x^{2}=2 x+3 \Longleftrightarrow y=x^{2}$ and $y=2 x+3$
Find $x$ at the intersection of these two graphs. as indicated in the figure (8.10).

| $\boldsymbol{x}$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\boldsymbol{y}$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |

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$S=\{-1,3\}$
Fig.(8.9)

$\mathrm{S}=\{-1,3\}$
Fig.(8.10)

Notice that the graphs of the four cases gave the same solution set $S=\{-\mathbf{1}, \mathbf{3}\}$.

### 8.2.The graphical method of solving in equalities:

Solving inequalities graphically depends up on the idea of determining the sign of the algebraic expression on both sides of the critical point using the critical value of the expression which is the value of the variable (variables) which makes the expression equals zero.

## Examples :

Determine the sign of each of the following expressions .
i) $\quad x+3$,
ii) $x-2$,
iii) $4-x$
iv) $2 x+5$

Solution : Determine first the critical value of each expression and then locat the critical point of each on the real line.
i) $x+3=0 \Longrightarrow x=-3 \Rightarrow$ critical value is -3 as indicated in the figure (8.11).


Fig.(8.11)
ii) $\quad x-2=0 \Longleftrightarrow x=2 \Longleftrightarrow$ critical value is 2 as indicated in the figure (8.12).


Fig.(8.12)
iii) $4-x=0 \Rightarrow x=4 \Rightarrow$ critical value is 4 as indicated in the figure the (8.13).


Fig.(8.13)
iv) $2 x+5=0 \Longrightarrow x=-\frac{5}{2} \Rightarrow$ critical value is $-\frac{5}{2}$ as indicated in the figure (8.14).


Fig.(8.14)
Notice that in the expression $\boldsymbol{a x}+\boldsymbol{b}$ if a is positive then the expression is positive on the right of the critical value and negative on its left. while the sign is the opposite to the above if a is negative ( ex , iii ).

### 8.2.1.Linear in equalities in one variable :

To solve graphically an in equality in the form $\boldsymbol{a} \boldsymbol{x}+\boldsymbol{b}_{>}^{<} \mathbf{0}, a \neq 0$ we put the in equality in the general form ( zero form ) then determine the critical value and local the critical point on the real line as shown in the following examples :

## Examples :

Solve graphically each of the following in equality
i) $\quad x+2 \geq 0$
ii) $x-5<0$
iii) $3 x+4>0$
iv) $2 x-3<3 x+5$

## Solution:

i) $\quad x+2 \geq 0$ put $x+2=0 \Longrightarrow x=-2$ ( critical valuet ) as indicated in the figure (8.15).


Fig.(8.15)
$\therefore$ solution set $S=[-2, \infty]$.
ii) $\quad x-5<0$, put $x-5=0 \Rightarrow x=5$ ( critical values ) as indicated in the figure (8.16).


Fig.(8.16)
$\therefore$ solution set $S=(-\infty, 5)$.
iii) $3 x+4>0$, put $3 x+4=0 \Longleftrightarrow x=-\frac{4}{3}$ (critical value ) as indicated in the figure (8.17).


Fig.(8.17)
$\therefore$ Solution set $S=\left(-\frac{4}{3}, \infty\right)$.
iv) $2 x-3<3 x+5$ put the in equality in the general form, then

$$
\begin{aligned}
& 2 x-3-3 x-5<0 \Longleftrightarrow-x-8<0 \\
& \text { put }-x-8=0 \Longleftrightarrow-x=8 \Longleftrightarrow x=-8 \quad(\text { critical value }) \text { as }
\end{aligned}
$$ indicated in the figure (8.18).



Fig.(8.18)
$\therefore$ the solution set is $S=(-8, \infty)$.
Note : we can put $-x-8<0$ in the form $x+8>0$ by multiplying by ( $1)$ then graph is as shown in the figure (8.19).


Fig.(8.19)
$\therefore S=(-8, \infty)$ which is the same as above.
Note: notice that the graphical method in this solution is more difficult than the algebraic (isolating showen in 3.4.1)

### 8.2.2. Quadratic inequatities in one variable .

To solve graphically a quadratic in equality in the form:
$\mathbf{a} \boldsymbol{x}^{2}+\mathbf{b} \boldsymbol{x}+\mathbf{c}_{>}^{<_{>}} \mathbf{0}, \mathbf{a} \neq \mathbf{0}$.
Put the in equality in the general form (zero form) then factorize the algebraic expression and determine the critical value for each factor and locate its critical point on the real line as shown in the following examples.

## Examples:

Solve graphically each of the following in equalities :
i) $x^{2}-x-6 \geq 0$
ii) $2 x^{2}-3 x+2<0$
iii) $x^{3}+2 x^{2}-3 x>0$

## Solution:

Factorize each of the expressions and determine the critical values of each as shown in the figure (8.20).
i)

$$
x^{2}-x-6 \geq 0 \Longleftrightarrow(x-3)(x+2) \geq 0
$$



Fig.(8.20-a)
$\therefore$ solution set is $(-\infty,-2] \cup[3, \infty)$
ii) $\quad 2 x^{2}-3 x+2<0 \quad(2 x-1)(x-2)<0$


Fig.(8.20-b)
$\therefore$ solution set is $\left(\frac{1}{2}, 2\right)$.
iii) $x^{3}-2 x^{2}-3 x>0 \Longrightarrow x\left(x^{2}-2 x-3\right)>0$

$$
\therefore x(x+1)(x-3)>0
$$



Fig.(8.20-c)
$\therefore$ solution set is $(-1,0) \cup(3, \infty)$.

### 8.2.3. Linear in equalities in two variables:

To solve graphically a linear in equality ( first degree ) in two variables in the form $a x+b y+c \geq 0$, we apply the following steps :

1) Put the inequality in the general form (zero form ) $a x+b y+$ $c \geq 0$.
2) On a graph paper plot the graph of the line $a x+b y+c=0$ notice that:
i) This line divides the co-ordinate plane into two parts called half-planes.

One to the right of the line and the other to the left of the line, this is if the line is not herizontal, as shown in the figure (8.21).


Fig.(8.21)
ii) The points on the straight line $\boldsymbol{a x}+\boldsymbol{b y}+\boldsymbol{c}=\mathbf{0}$, each makes the expression $\boldsymbol{a x}+\boldsymbol{b y}+\boldsymbol{c}$ equals zero.
iii) The straight line $\boldsymbol{a x}+\boldsymbol{b y}+\boldsymbol{c}=\mathbf{0}$ act as ciritical line to the expression $\boldsymbol{a x}+\boldsymbol{b y}+\boldsymbol{c}$.
iv) The points on one side of the line each makes the expression greater than zero and the points on the other side of the line each makes the expression less than zero.
3) Determining the solution set of the requied inequality which is the set of points which make the inquality true is done by identifying the halfplane on one of the sides of the line which its points make this inequality true, this is done by chosing a point in one of the sides of the line ( let it be the origin $(0,0)$ ) and by substitution in the expresion $\boldsymbol{a x}+\boldsymbol{b} \boldsymbol{y}+\boldsymbol{c}$ we can find out if the value of this expression is less than or greater than zero.

## Examples:

1) Solve graphically the inequality $5 x+y-3 \geq 0$.

Solution : the graph of the line $5 x+y-3=0$ is plotted and substitution by the origin $(0,0)$ in the expression we get:
$5 x+y-3 \Longleftrightarrow 0+0-3=-3$ which is less than zero Thus $5 x+y-3>0$ on the right of the line and $5 x+y-3<0$ on the left of the line and $5 x+y-3=0$ on the line, thus the solution set of the inequality $5 x+y-3 \geq 0$ is the set of points in the shaded are on the right of the line. See the figure (8.22).
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Fig.(8.22)
2) Discuss graphical $S_{1} \cap S_{2}$ if
$S_{1}=\{(x, y): 2 x-y+1<0\}$
$S_{2}=\{(x, y): x+y-3 \leq 0\}$

## Solution:

Figure (8.23) shows the intersection of these two sets which are the solution sets of the inequalities associated with them.
The crossed shadded area is the required intersection set.

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Terminology and Definitionsfor School Mathematics Concepts مصططلحات وتعاريف مفاهيم الرياضيـات المدرسيـة

Fig.(8.23)

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## السيرة الذاتية للموولْفين

الولّفالأول

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