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ATANASSOV'S INTUITIONISTIC FUZZY TOPOLOGY BASED ON THE SPACE OF INTUITIONISTIC FUZZY REAL NUMBERS (IFRN)

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Abstract
This study presents a new perspective on intuitionistic fuzzy topology through the integration of intuitionistic fuzzy real numbers and universal sets. Using theoretical foundations and operations developed by Atanassov and extended by Davvaz and others, we construct a topological framework that supports generalized fuzzy reasoning. The research introduces definitions and algebraic structures for intuitionistic fuzzy real numbers, explores topological convergence and continuity, and establishes their role in defining new fuzzy functions. This framework enhances expressive power in handling uncertainty.

Keyword: Intuitionistic fuzzy sets, Fuzzy real number, Fuzzy topology space, Intuitionis-tic fuzzy element, Intuitionistic fuzzy point, Fuzzy function.

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التبولوجيا الضبابية الحدسية "لأتاناسوف" المعتمدة على فضاء الأعداد الحقيقية الضبابية الحدسية

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الملخص

تقدم هذه الدراسة منظورًا جديدًا للتبولوجيا الضبابية الحدسية من خلال دمج الأعداد الحقيقية الضبابية الحدسية مع المجموعات الشاملة الضبابية .وبالاعتماد على الأسس النظرية التي وضعها أتاناسوف، والتي طُوّرت لاحقًا بواسطة دافاز وآخرين، تم بناء إطار تبولوجي يدعم الاستدلال الضبابي المعمم .تتضمن الورقة تعريفات وهياكل جبرية للأعداد الضبابية الحدسية، وتستكشف خصائص التقارب والاستمرارية في هذه الفضاءات، مع بيان دورها في تعريف الدوال الضبابية الجديدة .يساهم هذا الإطار في تعزيز القدرة التعبيرية لمعالجة حالات عدم اليقين.

الكلمات المفتاحية: المجموعات الضبابية الحدسية، العدد الحقيقي الضبابي، الفضاء التبولوجي الضبابي، العنصر الضبابي الحدسي، النقطة الضبابية الحدسية، الداله الضبابية،

1. INTRODUCTION

Atanassov in [1], [2] defined the concept of intuitionistic fuzzy theory and introduced new operations over the intuitionistic fuzzy sets. Then from that time, the theories of intuitionistic fuzzy mathematics have been developed in many applications, because the problem in fuzzy theory is how to carry out the ordinary notions of fuzzy theory.

Atanassov (1986), generalized the concept of the fuzzy set of Zadeh [3], as A= $\{<x, U_A(x), V_A(x)>: x \in X\}$, U_A, V_A are functions from X to [0,1], and defined the degree of membership and non-membership respectively of $x \in X$ in A (intuitionistic fuzzy subset of X), such that;

$$0 \leq U_A(x) + V_A(x) \leq 1, \quad V_A(x) = 1 - U_A(x).$$

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Rosenfeld [4] introduced the concept of fuzzy sets in the context of group theory and formulated the notion of fuzzy subgroups of a group.
Davvaz at el [5], [6], remarked on the absence of the intuitionistic fuzzy universal set and this absence has a strong effect on the fuzzy theory and intuitionistic fuzzy system.
The absence of the fuzzy universal sets watch introduced by Dib in [7] has a strong effect on the fuzzy theory; he replaced the concept of the universal set to define the notation of fuzzy space. The notion of intuitionistic fuzzy universal sets has been introduced by Davvaz et al. [8] in 2013, such that Davvaz and Karima, presented a formulation of intuitionistic fuzzy universal sets as a direct generalization of intuitionistic fuzzy space, such that an into intuitionistic fuzzy universal set $X^{\times I \times I}$ is the set of all ordered pairs $\{x\} \times I \times I$, $x \in X$, where $\{x\} \times I \times I = \{(x, r, s): r, s \in I\}$, hence an intuitionistic fuzzy sub universal set is the collection of all ordered pairs $(x, u_x, v_x), x \in U_0$ for some U_0 in X, and u_x *and* v_x are subsets of *I*.
We recall some definitions, which will be used in this paper:

***X*:** for a non-empty field of real numbers ℝ,

***I*:** the closed interval of ℝ,

***L*:** the Lattice [0, 1] × [0,1], such that, if (r_1, r_2), (s_1, s_2) are two open intervals and belong to I with partial order introduced by

1. $(r_1, r_2) \leq (s_1, s_2)$ *if* $r_1 \leq s_1, r_2 \leq s_2$ *such that* $s_1 \neq 0$, $s_2 \neq 0$,
2. $(s_1, s_2) = (0, 0)$ *if* $s_1 = 0$ *or* $s_2 = 0$.
3.

In [9], Dib introduced the concept of fuzzy real number in fuzzy space as a fuzzy element of any fuzzy subspace of fuzzy space (R, I), It has the form (r, u_r), where u_r contains at least one number more than zero. As Dib in [10], on the space of fuzzy real numbers, to introduce the concept of a fuzzy topology, which is similar to topology in the usual case. Dib at el [11], using the concept of fuzzy space, they are going parallel to the ordinary case for introduce the fuzzy functions from the fuzzy space (X, I) to the fuzzy space (Y, I) as $\underline{F} = (F, f_x)$; where F: X ⟶ Y, f_x $(\forall x \in X)$ is a family of into membership function f_x: [0,1] ⟶ [0,1], satisfying the following conditions:

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i. f_x is non-decreasing function on [0,1],
ii. $f_x(0) = 0$, and $f_x(1) = 1$
iii.

Ahmed Sedky [12], introduced a short surrey of fuzzy sets. Also, he introduced the notions of lattice-valued topology and studied some properties of L-fuzzy topological spaces. Zahan [13], classified the fuzzy sets and topological spaces, and introduced the relation between elements of them. Subhankar at el [14], used Pythagorean fuzzy to determine the topological relation between two spatial objects, and studied of Pythagorean fuzzy topological spaces by introducing core, fringe, outer, regular open set, regular closed set, double connectedness and homeomorphism in Pythagorean fuzzy environment. Ameziane at el. [15], defined the sequential fuzzy closure and sequential fuzzy interior of fuzzy subsetsof [0.1].

2. INTUITIONISTIC FUZZY REAL NUMBER

The intuitionistic fuzzy universal set $R^{I \times I}$ replaces $\mathbb{R}$ (the set of real numbers) in the fuzzy case. Therefore, the intuitionistic fuzzy elements which in the form $(\{r\} \times I \times I)$ replace of r. Then in the fuzzy case, the fuzzy real numbers are the elements of sub universal sets of the fuzzy universal set $R^{\times I}$ or fuzzy elements $(\{r\}, [0,1])$ of $\mathbb{R}$. Then, in the fuzzy theory, we defined and introduced the following;

Definition 1:

An intuitionistic fuzzy element of any intuitionistic fuzzy sub-universal set $U^{\times I \times I}$ of $\mathbb{R}^{\times I \times I}$ is intuitionistic fuzzy real number, i.e., the intuitionistic fuzzy numbers have the form (r, u_r, v_r), such that u_r is a membership function and v_r is a non-membership
function and contains at least one number more than zero. The multiplication and the
addition of intuitionistic fuzzy real numbers are intuitionistic fuzzy binary hyperoperations $\underset{\sim}{F} = < \underline{F}, \underline{f}_{xy}, \overline{f}_{xy} > : R^{\times I \times I} \times R^{\times I \times I} \rightarrow P^{*}(R^{\times I \times I})$, on intuitionistic fuzzy universal set $\mathbb{R}^{\times I \times I}$ consists of three parts as the following:

i. In the ordinary case, the intuitionistic fuzzy binary hypeoperations (+ or •) on the intuitionistic fuzzy real number $\mathbb{R}$ based on intuitionistic fuzzy universal set $R^{\times I \times I}$, with the corresponding ordinary binary operation $\underline{F}$ of $\mathbb{R}$

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ii. In the fuzzy case, the intuitionistic fuzzy binary hyperoperation consists of the comembership functions $f\varkappa$ on the fuzzy universal set $R^{\times I}$, I= [0,1].

iii. In the intuitionistic fuzzy case, the intuitionistic fuzzy hyperbinary operation consists of the co-membership functions $\underline{f}_{xy}$ and conon-membership functions $\overline{f}_{xy}$ on intuitionistic fuzzy universal set $R^{I \times I}$.

In this case, a uniform comembership function and a conon-membership function are chosen as follows:

Definition 2:

We define the intuitionistic fuzzy binary hyperoperations $\pm = <\pm, \underline{\Psi}_{xy}, \overline{\Psi}_{xy}>$ and $: = <:, \underline{\phi}_{xy}, \overline{\phi}_{xy}>$, such that +, : are the ordinary operations **(**addition (minus) and multiplication on ℝ**)**, but it's not mentioned else. Then we can consider $\underline{\Psi}_{xy} = \underline{\phi}_{xy} = r\wedge s$ and $\overline{\Psi}_{xy} = \overline{\phi}_{xy} = r \vee s$.

According to the above definition, we have the following:

$$(\varkappa, \underline{u}_x, \bar{u}_x) \pm (y, \underline{u}_y, \bar{u}_y) = (\varkappa \pm y, \underline{u}_x \wedge \underline{u}, \bar{u}_x \vee \bar{u}_y),$$
$$(\varkappa, \underline{u}_x, \bar{u}_x) : (y, \underline{u}_y, \bar{u}_y) = (\varkappa : y, \underline{u} \wedge \underline{u}_y, \bar{u}_x \vee \bar{u}_y).$$

Remark 3.

Let S_1, S_2 are two intuitionistic fuzzy subsets of X, then by the intuitionistic fuzzy hyperoperations**,**

$$(s_1 \pm s_2)(\varkappa) = \vee_{s,t \in x} (\underline{s}_1(t) \wedge \underline{s}_2(s), \bar{s}_1(t) \vee \bar{s}_2(s)),$$
$$(s_1 : s_2)(\varkappa) = \vee_{s,t \in x} (\underline{s}_1(t) \wedge \underline{s}_2(s), \bar{s}_1(t) \vee \bar{s}_2(s)).$$

Where, $x = t \pm s \ (in\ the\ first\ equation)$,
$x = t : s \ (in\ the\ schond\ equatuin)$.

Theorem 4:

For any onto comembership function and co non-membership function where $< X, \pm, : >$ is an isomorphic intuitionistic fuzzy field to the fuzzy real field $< R^{\times I}, \pm, : >$ by the correspondence between the intuitionistic fuzzy real number $\{x\} \times I \times I$ and the fuzzy real number $\{x\} \times I \times I$ as following; $\{x\} \times I \times I \Leftrightarrow \{x\} \times I$.

Proof: *The proof follows from the definition of intuitionistic fuzzy operations.*

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Lemma 5

If $a = < \varkappa, \underline{u}_x, \overline{u}_x>$, $b = < y, \underline{u}_y, \overline{u}_y>$ and $C = < z, \underline{u}_z, \overline{u}_z>$ are intuitionistic fuzzy real numbers, then we have;

1- $a \pm (b \pm c) = (a \pm b) \pm C$,
2- $a: (b:c) = (a:b) : C$,
3- $a \pm b = b \pm a$,
4- $a: b = b : a$,
5- $a: (b \pm c) = a: b \pm a: c$,
6- $(0, \underline{u}_x, \overline{u}_x) + a = a = a + (0, \underline{u}_x, \overline{u}_x)$,
7- $(1, \underline{u}_x, \overline{u}_x) : a = a = a : (1, \underline{u}_x, \overline{u}_x)$,
8- $(-\varkappa, \underline{u}_x, \overline{u}_x) + (\varkappa, \underline{u}_x, \overline{u}_x) = (0, \underline{u}_x, \overline{u}_x)$,
9- $a. (\frac{1}{a}, \underline{u}_x, \overline{u}_x) = (1, \underline{u}_x, \overline{u}_x); \varkappa \neq 0$.

We will choose the intuitionistic fuzzy hyperoperations on intuitionistic fuzzy universal sets as follows:

$\underline{\Psi}_{xy}(r,s), \underline{\emptyset}_{xy}(r,s) = r \wedge s$ and $\overline{\Psi}_{xy}(r,s), \overline{\emptyset}_{xy}(r,s) = r \vee s$,

where the co-membership functions of intuitionistic fuzzy addition $\underline{\Psi}_{xy}(r,s)$,
$\underline{\emptyset}_{xy}(r,s)$ and conon-membership functions of intuitionistic fuzzy multiplication $\overline{\Psi}_{xy}(r,s)$, $\overline{\emptyset}_{xy}(r,s) = r \vee s$ as (uniform functions).
As the intuitionistic fuzzy real numbers have an inverse relative to intuitionstic fuzzy multiplication and intuitionistic fuzzy addition. But in general, we can choose the comembership functions $\underline{\Psi}_{xy}$, $\underline{\emptyset}_{xy}$ and the cononmembership functions $\overline{\Psi}_{xy}$, $\overline{\emptyset}_{xy}$ as uniform and non-uniform (different) as follows:

$\underline{\Psi}_{xy}(r,s) = r \wedge s$, $\underline{\emptyset}_{xy}(r,s) = r + s$ and $\overline{\Psi}_{xy}(r,s) = r: s$, $\overline{\emptyset}_{xy}(r,s) = r \vee s$.

Since we cannot say that each intuitionistic fuzzy real numbers take the form
$< x, \{0, r\}, \{s, 1\}>$ such that $r \neq 1$, $s \neq 0$, have multiplicative inverse. Then, the relationship that is satisfied by intuitionistic fuzzy real numbers depends on the given co-membership functions and co-nonmembership functions.
Here in the fuzzy theory, we choose an ordinary set of natural numbers where count a finite number of elements of intuitionistic

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fuzzy universal sets and intuitionistic fuzzy subuniversal sets using the intuitionistic fuzzy real numbers in the form:

$$< x, \{0, r\}, \{s, 1\}> = < n, r, s >; 0 < r \leq 1, 0 \leq s < 1, n \in \aleph,$$

Experiments are conducted to determine the number of elements in intuitionistic fuzzy universal sets and intuitionistic fuzzy sub-universal sets.

Let $X^{I \times I}$, represent an intuitionistic fuzzy universal set and let μ denote the measure defined on the closed unit interval $I = [0,1]$.

Intuitionistic fuzzy real number (r, I, I) is an intuitionistic fuzzy element of the intuitionistic fuzzy universal set $R^{I \times I}$.

Example 6: If we have the numbers of balls, then it is an ordinary subset of intuitionistic fuzzy universal set that contains some numbers of these balls. Then some of these balls are represented by n (a natural number) and embedded in the intuitionistic fuzzy real numbers $n^{I \times I} = < n, I, I >$. However, the intuitionistic fuzzy subset encompasses elements, which correspond with the parts of balls.

Example 7: Let $\{x^{\times [0,\frac{3}{4}] \times [\frac{3}{4},1]}\}$ be an intuitionistic fuzzy sub-universal set, which encompasses one intuitionistic fuzzy element as well as the intuitionistic fuzzy sub universal set $\{x^{I \times I}\}$. Then $x^{\times [0,\frac{3}{4}] \times [\frac{3}{4},1]}$ must be defined by an intuitionistic fuzzy number and it's less than the value associated with count $x^{\times I \times I}$, the intuitionistic fuzzy element $x^{\times I \times I}$, contains all possible comembership value and co-nonmembership value corresponding to the element $\varkappa$ in the context of intuitionistic fuzzy universal set $x^{\times I \times I}$, then that $x^{\times I \times I}$ must be counted by <x,1,1>, but $x^{\times [0,\frac{3}{4}] \times [\frac{3}{4},1]}$ must be counted by $< 1, \mu\, [0,3/4], v\, [3/4, 1] > = <x, 0,1>$. Therefore, the intuitionistic fuzzy element $< x, \Delta_1, \Delta_2 >$; where Δ_1, Δ_2 are μ- measurable subsets of [0,1], which must be counted by $<1, \mu\, (\Delta_1), v\, (\Delta_2) >$ if $\mu\, (I) = 0, \nu(I) = 1$.

3. INTUITIONISTIC FUZZY TOPOLOGY ON SPACE OF INTUITIONISTIC FUZZY REAL NUMBER (IFRNs)

The notion of intuitionistic fuzzy topology on intuitionistic fuzzy real numbers in this case corresponds to the used topology. Then, by using the concept of intuitionistic fuzzy universal sets and the family of intuitionistic fuzzy open intervals, we generate the

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intuitionistic fuzzy topology on the space of intuitionistic fuzzy real numbers.

Remark 8: The operations on the intuitionistic fuzzy subuniversal sets are $\vee$ (supremum), and $\wedge$ (infimum) , but the fuzzy operations in fuzzy universal sets are $\cup$ (union) and $\cap$ (intersection).

Definition 9: If $\mathbb{R}^{\times I \times I}$ is the space of an intuitionistic fuzzy universal set, then the sets U and V are open intuitionistic fuzzy intervals of τ (the family of intuitionistic fuzzy subuniversal sets of $\mathbb{R}^{\times I \times I}$), then the family τ of $\mathbb{R}^{\times I \times I}$ is called an intuitionistic fuzzy topology on $\mathbb{R}^{\times I \times I}$, if τ satisfies the following conditions:

1. $\mathbb{R}^{\times I \times I}$ and ϕ are intuitionistic fuzzy subuniversal sets of τ,
2. $U \wedge V,$ is intuitionistic fuzzy subuniversal sets of τ, for two open intuitionistic fuzzy intervals of τ,
3. $\bigvee_{U_{i \in \tau_i}} U_i$ is in τ $\forall$ $U_i \subset \tau$.

Therefore, the open universal sets of the intuitionistic fuzzy topology $(\mathbb{R}^{\times I \times I}, \tau)$ are the union and intersection of uniform intuitionistic fuzzy sub universal sets, then
$]U, \{0\} \wedge F_1, F_2 \vee \{1\}[\ ; F_1 = U_i \cap [0,1], F_1 = U_i \cap [0,1],$
where U, U_i , F_1, and F_2 are open intervals of the real space R.

Remark 10:

I. For each intuitionistic fuzzy topology $(\mathbb{R}^{\times I \times I}, \tau)$ depending on the space of intuitionistic fuzzy real numbers, there is a classical topology
$\tau_{[0,1]} = \cap_{r \in R}\ \tau_{[0,1]}$.

II. Since the family τ is closed under ($\vee$ arbitrary unions), and the family τ is closed under finite intersection, then it is the family of topologies on $\mathbb{R}$.

Example 11: Let $\mathbb{R} = \{1,2,3\},\ \tau = \{\emptyset, \mathbb{R}^{\times I \times I}, \Delta_1, \Delta_2\}$ and $\tau_{\mathbb{R}}$ under finite intersection is not closed,

$\Delta_1 = \{ (1, [0,1]), (2, \{0\} \cup \frac{1}{2})\},$

$\Delta_2 = \{ \left(2, \{0\} \cup \frac{1}{4}\right), (3, [0,1]) \}.$

It is clear $\tau_{\mathbb{R}}$ under finite intersection is not closed.

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Definition 12:

The intuitionistic fuzzy sub universal set is one of the open intuitionistic fuzzy sub universal sets of a given topology and can be written in the form

$U = \{x^{\times\{0\}UF1\times F2U\{1\}}$: $F_1 = (r_1(x), r_2(x))$, $F_2 = (s_1(x), s_2(x))$, $x \in (a, b) \ni a, b \in R\}$, such that (a, b), $(r_1(x), r_2(x))$, $(s_1(x), s_2(x))$ are open intervals in [0,1] for all $x \in (a, b)$. Then U can be written in the form

$U = \bigvee_{\tau} \{ x^{\times\{0\}UF1\times F2U\{1\}};\ x \in U_i^n\}$, where U^n_τ are open intervals, and for a given n, we have $\bigvee_\tau U^n_i$ is the form (a, b), where $U^n_i < \frac{1}{n}$ $\forall n \in N$,

$x \in U\tau$, then

$$U = \bigcup_{n=1}^{\infty} U_\tau \{(x, \{0\} \cup\ ]\ max\ r_1(x), min\ r_2(x)\ [,\ ]\ max\ s_1(x), min\ s_2(x)\ [\ \cup \{1\})\}$$
$$= \bigcup_{n=1}^{\infty} U_i\ (U^n{}_i, (\{0\} \cup\ ]\ max\ r_1(x), min\ r_2(x)\ [,\ ]\ max\ s_1(x), min\ s_2(x)\ [\ \cup \{1\})\ ) \quad ;\ for\ all\ x \in U_i^n.$$

Then we can say that C (the intuitionistic fuzzy subuniversal set) is closed if the complement is open, i.e., $(C \subseteq f^C)$, then the closed interval is a closed intuitionistic fuzzy subuniversal set, and can be written in the form $C = < [a,b], \{0\} \cup [r_1,r_2], [s_1,s_2] \cup \{1\} >$; where $r_1 \leq r_2 \leq 1$, $0 \leq s_1 \leq s_2$.

Remark 13:

I. An open intuitionistic fuzzy subuniversal set and the largest containing U is the interior of intuitionistic fuzzy subuniversal set U, and is denoted by U^0.

II. The closure of intuitionistic fuzzy subuniversal set U, denoted by $\bar{U}$ such that $U \subset \bar{U}$.

III. The sequence of simple intuitionistic fuzzy numbers $((x_n, \{0\} \vee F_{n1}, F_{n2} \vee \{1\}) \longrightarrow ((x_0, \{0\} \vee F_{01}, F_{02} \vee \{1\})$, if every open intuitionistic fuzzy universal set containing the intuitionistic fuzzy $((x_0, \{0\} \vee F_{01}, F_{02} \vee \{1\})$, contains all elements of this sequence except a finite number.

IV. For every simple intuitionistic fuzzy real number, there is a sequence of open intuitionistic fuzzy intervals $(]\ x_0 - \frac{1}{n},\ x_0 + \frac{1}{n}\ [,\ \{0\} \cup]\ F_{01} - \frac{1}{n}, F_{01} + \frac{1}{n}\ [\ ,\ ]\ F_{02} - \frac{1}{n}, F_{02} + \frac{1}{n}\ [\cup\{1\})$; where $0 < F_{01} - \frac{1}{n}, F_{01} + \frac{1}{n} \leq 1$ and $0 < F_{02} - \frac{1}{n}, F_{02} + \frac{1}{n} \leq 1$, (for sufficiently large n).

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Theorem 14:

If $U_n(x) = \,]\, x_n, \{0\} \cup F_{n1}, F_{n2} \cup \{1\}[$ are (IFRNs), then the sequence of $U_n(x)$ is converges to $< x_0, \{0, F_{01}\}, \{F_{02}, 1\} > = U_o(x)$ (simple intuitionistic fuzzy subuniversal set) by convergent sequence i*ff* the sequence for every $x_0, x_n \in \mathbb{R}$, where ($x_n \to x_0$, $F_{n1} \to F_{01}$, $F_{n2} \to F_{02}$),

Then, by using the above theorem, we get algebra properties of the convergent sequences of $U_n(x)$, as follows;

If $U_n(x) \longrightarrow U_o(x)$, $U_n(y) \longrightarrow U_o(y)$, then we have,

$<x_n, \{0\} \cup r_{n1}(x), r_{n2}(x) \cup \{1\}> + <y_n, \{0\} \cup s_{n1}(y), S_{n2}(y) \cup \{1\}> \longrightarrow <x_o, \{0, r_{O1}(x_o)\}, \{r_{O2}(x_o), 1\} > + < y_o, \{0, s_{o1}(y_o)\}, \{s_{o2}(y_o), 1\} >$,

And

$<x_n, \{0\} \cup r_{n1}(x), r_{n2}(x) \cup \{1\}>: < y_n, \{0\} \cup s_{n1}(y), s_{n2}(y), \cup \{1\}> \longrightarrow <x_o, \{0, r_{O1}(x_o)\}, \{r_{O2}(x_o), 1\} >: <y_o, \{0, s_{o1}(y_o)\}, \{s_{o2}(y_o), 1\} >$,

$<x_n, \{0\} \cup r_{n1}(x), r_{n2}(x) \cup \{1\}> / <y_n, \{o\} \cup s_{n1}(y), s_{n2}(y), \cup \{1\}> \longrightarrow < x_o, \{0, r_{O1}(x_o)\}, \{r_{O2}(x_o), 1\} > / <y_o, \{0, s_{o1}(y_o)\}, \{s_{o2}(y_o), 1\} >$; $y_n \neq 0$, $y_o \neq 0$.

However, if we have $]\, x_n, \{0\} \cup \{r_n\}, \{s_n\} \cup \{1\} [\longrightarrow]\, x_o, \{0, r_{O1}\}, \{s_{o1}, 1\} [$ and

$]\, y_m, \{0\} \cup \{r_m\}, \{s_m\} \cup \{1\} [\longrightarrow]\, y_o, \{0, r_{o2}\}, \{s_{o2}, 1\} [$; $0 < r_i \leq 1$, $0 \leq si \leq 1$,

then, we have the intuitionistic fuzzy sequences as follows:

$$]\, x_n, \{0\} \cup \{r_n\}, \{s_n\} \cup \{1\} [+]\, y_n, \{0\} \cup \{r_m\}, \{s_m\} \cup \{1\} [$$

$$=]\, x_n + y_m, \{0\} \cup \underline{f}_{nm}(r_n, r_m), \overline{f}_{nm}(s_n, s_m) \cup \{1\} [$$

$$\longrightarrow]\, x_o + y_o [, \{0\} \cup \underline{f}_{nm}(r_{o1}, r_{o2}), \overline{f}_{nm}(s_{o1}, s_{o2}) \cup \{1\} [$$

and

$$]\, x_n, \{0\} \cup \{r_n\}, \{s_n\} \cup \{1\} [\;:\;]\, y_m, \{0\} \cup \{r_m\}, \{s_m\} \cup \{1\} [$$

$$=]\, x_n : y_m, \{0\} \cup \underline{\psi}_{nm}(r_n, r_m), \overline{\psi}_{nm}(s_n, s_m) \cup \{1\} [$$

$$\longrightarrow]\, x_o : y_o, \{0\} \cup \underline{\psi}_{nm}(r_{o1}, r_{o2}), \overline{\psi}_{nm}(s_{o1}, s_{o2}) \cup \{1\} [$$

The sequences are convergent if

$$\underline{f}_{xy}(r_n, r_m), \overline{f}_{xy}(s_n, s_m), \underline{\psi}_{xy}(r_n, r_m), \overline{\psi}_{xy}(s_n, s_m).$$

are continuous functions on [0,1] and $\mathbb{R}$:

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Definition 15:

If $U = \{<x, u_x, v_x> ; x \in Uo\}$ is an intuitionistic fuzzy sub-universal set, then the point $U_o = < x_o, \{0,r_o\}, \{s_o,1\} >$ is called a limit point of U if there is a sequence of simple intuitionistic fuzzy numbers

$< x_n, \{0\} \cup \{r_n\}, \{s_n\} \cup \{1\}>$ in U, that is $x_n \in Uo$, r_n , $s_n \subseteq U_x$, $0< r_n <1$, $0 < S_n \; 1$,as that U converges to Uo.

Definition 16:

An intuitionistic fuzzy sub universal set U of intuitionistic fuzzy real number $R^{X[0,1]x[0,1]}$ is said to be a compact intuitionistic fuzzy sub-universal set if for every open covering of U, there exists a finite sub-covering.

Remark 17:

1- Every closed intuitionistic fuzzy interval contains all limit points,
2- Every different intuitionistic fuzzy points $\propto_1, \propto_2, \dots \dots$ exist closed intervals $C_{\alpha_i}; \propto_i \in C^o_{\propto\iota}$; $\iota = 1,2$ and $C_{\propto\iota} \cap C_{\propto 2} = \emptyset$,
3- For each bounded sequence of simple intuitionistic fuzzy real numbers (IFRNs), there exists a convergent sub sequence,
4- Each closed, bounded intuitionistic fuzzy interval is compact.
5- Every intuitionistic fuzzy sub universal set is compact if every sequence of intuitionistic fuzzy real numbers has a convergent sub sequence.

Definition 18:

If the simple intuitionistic fuzzy real numbers are in the form, $x_n^{\times [0, rn] \times [Sn, 1]}$, then the sequence of intuitionistic fuzzy real numbers is converges to the limit point of intuitionistic fuzzy subuniversal set $x_o^{\{0,ro\}\times\{so,1\}}$, if for every $x_o^{\{0,\gamma o\}\times\{\propto o,1\}} \in x_o^{\{0,ro\}\times\{so,1\}}$ there exists a sequence of simple intuitionistic fuzzy numbers $x_n^{\times \gamma n \times \alpha n} \in x_n^{\times [0,rn\}\times\{sn,1\}}$ and it converges to $x_o^{\{0,\gamma o\}\times\{\propto o,1\}}$.

Then, directly from this definition we set the following;

Theorem 19:

The sequence of intuitionistic fuzzy counting the numbers in the form, $x_n^{\times\propto n\times xn}$ converges to $x_o^{\times ro \times so} \Leftrightarrow x_n \rightarrow x_o, \gamma_n \rightarrow r_n, \propto_n \rightarrow S_o$.

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4. CONTINUITY OF INTUITIONISTIC FUZZY FUNCTIONS

Davvaz et al. (2013) defined the concept of intuitionistic fuzzy function on an intuitionistic fuzzy universal set $X^{\times[0,1]\times[0,1]}$ as:
$\underset{\sim}{F}: X^{\times[0,1]\times[0,1]} \times X^{\times[0,1]\times[0,1]} \to X^{\times[0,1]\times[0,1]}$,
with co-membership functions $\underline{f}_{xy}$ and conon-membership functions $\overline{f}_{xy}$ satisfying:

1− $\underline{f}_{xy}(r,s) \neq 0$ if $r \neq 0, s \neq 0$ and $\overline{f}_{xy}(w,z) \neq 1$ if $w \neq 1, z \neq 1$.
2− $\underline{f}_{xy}$ and $\overline{f}_{xy}$ are onto. That is, $\underline{f}_{xy}([0,1]\times[0,1]) = [0,1]$ and $\overline{\overline{f}}_{xy}(I \times I) = I$.

Definition 20:
The intuitionistic fuzzy functions

$\underset{\sim}{F} = < \underline{F}, \underline{f}_{xy}, \overline{f}_{xy} > : X^{\times[0,1]\times[0,1]} \times X^{\times[0,1]\times[0,1]} \to X^{\times[0,1]\times[0,1]}$
are continuous intuitionistic fuzzy functions if $\underline{F}^{-1}$ (O) is open intuitionistic fuzzy sub universal set (O is an open intuitionistic fuzzy subuniversal set of $X^{\times[0,1]\times[0,1]}$,
$\underset{\sim}{F}^{-1}(\vee_i o_i) = \vee_i \underset{\sim}{F}^{-1}(o_i)$; X is the field of real numbers. Then we introduce that the intuitionistic fuzzy functions $\underset{\sim}{F}$ are continuous if for any open intuitionistic fuzzy interval $\underset{\sim}{F}^{-1}$ is an open intuitionistic fuzzy sub universal set.

Theorem 21:
The intuitionistic fuzzy function $\underset{\sim}{F} = < \underline{F}, \underline{f}_{xy}, \overline{f}_{xy} > : X^{\times I\times I} \times X^{\times I\times I} \to X^{\times I\times I}$ is continuous intuitionistic fuzzy function i*ff* $\underline{F} : X \times X \to X$ is ordinarily continuous and co-membership function $\underline{f}_{xy}(r,s)$ and conon-membership function $\overline{f}_{xy}(r, s)$ are continuous in *x*, *r* and *s*, for every *r*, *s* ∈ [0,1] and *x* belong to *X* (the field of real numbers).

Proof:⟹
(**i**) ⟹: If we have an intuitionistic fuzzy point $(x_o, \{0,r\}, \{s,1\})$ in $X^{\times I\times I}$, and
$\underset{\sim}{F} = < \underline{F}, \underline{f}_{xy}, \overline{f}_{xy} > : X^{\times I\times I} \times X^{\times I\times I} \to X^{\times I\times I}$ is intuitionistic fuzzy operation and its continuous intuitionistic fuzzy function, such that the intuitionistic fuzzy point has the image

$$< \underline{F}(x_o,y_o), \{0\} \vee \underline{f}_{xoyo}(r_{o1},s_{o1}), \overline{f}_{xoyo}(r_{o2},s_{o2}) \vee \{1\} >.$$

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Let $0 < \underline{f}_{xoyo} \leq 1$, $0 \leq \overline{f}_{xoyo} < 1$ and $< \delta_1, \delta_2 >$ is an intuitionistic fuzzy number, where $\delta_1, \delta \in R^+$ i.e., δ_1, δ_2 are real numbers, $0 < \delta_1 < \underline{f}_{xo\,yo}(r_{o1}, s_{o1}) < \overline{f}_{xo\,yo}(r_{o2}, s_{o2}) < 1 - \delta_2$, and let

$S_1 = < \underline{F}(x_o, y_o) - \delta_1, < \underline{F}(x_o, y_o) + \delta_1, \underline{f}_{xoyo}(x_o, y_o) - \delta_2, \overline{f}_{xoyo}(x_o, y_o) + \delta_2 >$ is an intuitionistic fuzzy point. Then $\underset{\sim}{F}^{-1}(s_1)$ is an open intuitionistic fuzzy sub universal set, and contains of the intuitionistic fuzzy point $< x_o, \{0, r_o\}, \{s_o, 1\} > \in X^{\times I \times I}$, hence there exists the form $<\varepsilon_1, \varepsilon_2>$, such that $\underset{\sim}{F}^{-1}(s_1)$ contains the intuitionistic fuzzy open interval S_2, that is,

$\underset{\sim}{F}(s_2) \subset S_1$, then for any; $x \in (x_o - \varepsilon_1, y_o + \varepsilon_1)$;
$S_2 = (x_o - \varepsilon_1, x_o + \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2)$,
$\underline{F}((x_o - \varepsilon_1, y_o + \varepsilon_1)) \subset \underline{F}(\underline{F}(x_o) - \delta_1, F(y_o) + \delta_1)$,
$\underline{f}_{xy}((\varepsilon_o - \varepsilon_2, \varepsilon_o, + \varepsilon_2)) \subset (\underline{f}_{xo\,yo}(\varepsilon_o) - s_2, \underline{f}_{xo\,yo}(\varepsilon_o) + \delta_2)$,
$\overline{f}_{xy}((\varepsilon_o - \varepsilon_2, \varepsilon_o + \varepsilon_2)) \subset (\overline{f}_{xoyo}(\varepsilon_o) - \delta_2, \overline{f}_{xo\,yo}(\varepsilon_o) + \delta_2)$.

Then, the ordinary function, $\underline{F}: X \times X \to X$, be a continuous at (x_o, y_o) and the functions $\underline{f}_{xy}(r, s)$, $\overline{f}_{xy}(r, s)$ are continuous relative to r, s.

(ii) $\Leftarrow$: Let the intuitionistic fuzzy function
$\underset{\sim}{F} = < \underline{F}, \underline{f}_{xy}, \overline{f}_{xy} > : X^{\times I \times I} \times X^{\times I \times I} \to X^{\times I \times I}$,
be a continuous relative to the usual topology on X (the field of the real numbers), and the functions $\underline{f}_{xy}(r,s)$, $\overline{f}_{xy}(r,s)$ are continuous at r,s, and X is the field of $\mathbb{R}$.
Now, consider the intuitionistic fuzzy open interval as $< x_1, y_1, x_2, y_2>$, and
$\underline{f}^{-1}_{xy}(]\,x_2, y_2[\,)$ and $\overline{f}^{-1}_{xy}(\,]\,x_2, y_2[\,)$ are open sets for all $\underline{F}(x,y) \in < x_1, y_1 >$,then,
$\vee_{x,y} < \underline{F}^{-1}\,]\,(x_1, y_1)\,[, \underline{f}^{-1}_{xy}]\,(x_2, y_2)\,[, \overline{f}^{-1}_{xy}\,]\,(x_2, y_2)\,[\,>$
Be an intuitionistic fuzzy open sub universal set, if we have
$\underset{\sim}{F}^{-1} < (x_1, y_1, x_2, y_2) >= \vee\ ((x, y), \underline{f}^{-1}_{xy}(x_1, y_1), \overline{f}^{-1}_{xy}(x_2, y_2))$
Hence by using the definition of the intuitionistic fuzzy binary operations on intuitionistic fuzzy real numbers, we can get the algebra properties of intuitionistic fuzzy functions as the follows:
Consider $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are two intuitionistic fuzzy binary operations on real numbers, such that $\underset{\sim}{F} = <\underline{F}, \underline{f}_{xy}, \overline{f}_{xy}>$ and $\underset{\sim}{G} = <\underline{G}, <\underline{g}_{xy}, \overline{g}_{xy}>$

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are intuitionistic fuzzy functions of the intuitionistic fuzzy real numbers, then we have

$$\underset{\sim}{F} \pm \underset{\sim}{G} = < \underline{F} \pm \underline{G}, \underline{f}_{xy} \wedge \underline{g}_{xy}, \overline{f}_{xy} \vee \overline{g}_{xy} >,$$

$$\underset{\sim}{F} : \underset{\sim}{G} = < \underline{F} . \underline{G}, \underline{f}_{xy} \wedge \underline{g}_{xy}, \overline{f}_{xy} \vee \overline{g}_{xy} >,$$

$$\underset{\sim}{F} / \underset{\sim}{G} = < \underline{F} / \underline{G}, \underline{f}_{xy} \wedge \underline{g}_{xy}, \overline{f}_{xy} \vee \overline{g}_{xy} >,$$

Corollary 22:

I. If intuitionistic fuzzy binary operations $\underset{\sim}{F}$ and $\underset{\sim}{G}$, are continuous intuitionistic fuzzy functions, then we get, $\underset{\sim}{F} \pm \underset{\sim}{G}$, $\underset{\sim}{F}.\underset{\sim}{G}$ *and* $\underset{\sim}{F}/\underset{\sim}{G}$ are continuous functions.

II. If we have that co-membership function $\underline{f}_{xy}(r,s)$ and conon- -membership function $\overline{f}_{xy}(r,s)$, $0 < r < s \leq 1$, are continuous functions relative to the topology universal set R, then we have two operations are continuous ($\underset{\sim}{F} + \underset{\sim}{G}$, $\underset{\sim}{F} \cdot \underset{\sim}{G}$), while $\underset{\sim}{F}/\underset{\sim}{G}$ is continuous if it's defined. But it is not defined in general.

Conclusions

1. In this study, we have built upon the foundational concepts of intuitionistic fuzzy sets, as introduced by Atanassov, and further developed through the notion of intuitionistic fuzzy universal sets and intuitionistic fuzzy real numbers.
2. By extending the classical fuzzy topological space, we have constructed a comprehensive framework for intuitionistic fuzzy topology based on intuitionistic fuzzy real numbers (IFRNs). This framework preserves the essential properties of fuzzy topology whi le offering greater flexibility and expressiveness through the use of co-membership and co-non-membership functions.
3. Our methodology involved defining intuitionistic fuzzy elements, binary hyper operations, and constructing topological spaces that reflect these generalized structures.
4. The paper also explored the algebraic and topological behavior of sequences and functions within these spaces, including their continuity and convergence properties.
5. The results confirm that intuitionistic fuzzy topology based on IFRNs forms a robust generalization of classical fuzzy topologies. It accommodates more nuanced representations of uncertainty and provides a fertile ground for further theoretical exploration and application in fields such as decision theory, information systems.

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