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Characterizing Solution of Multiobjective Non-Linear Programming with Rough Objective Functions and Fuzzy Parameter in the Constraints

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Abstract

This paper presents a solution to multiple objective non-linear programming (MONLP) problems with rough objective functions and constraints with fuzzy parameters, we offer an algorithm to solve this type of problem, where we deal with multiple objective functions first, then with the fuzziness of the constraint parameter Characterizing, second and third, we deal with the roughness of the objective function. So, the problem finally turns into a non-linear programming problem with crisp objective functions and constraints. Finally, an illustrative example is given to show the application of the algorithm.

Keywords: Multiobjective non-linear programming (MONLP), rough function, Fuzzy number, Ranking function.

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توصيف حل البرمجة غير الخطية متعددة الأهداف ذات دوال هدف استقرابية و معلمات ضبابية في القيود

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الملخص

تقدم هذه الورقة حلاً لمشاكل البرمجة غير الخطية متعددة الأهداف مع دوال هدف استقرائية (رافية) وقيود ذات معلمات ضبابية (فازية)، نقدم خوارزمية لحل هذا النوع من المشاكل، حيث نتعامل مع دوال هدف متعددة أولاً، ثم مع ضبابية معلمة القيد ثانيا. وثالثا نتعامل مع استقرابية دالة الهدف. لذا، تتحول المشكلة أخيرًا إلى مشكلة برمجة غير خطية مع دالة هدف وقيود واضحة. أخيرًا، يتم تقديم مثال توضيحي لإظهار تطبيق الخوارزمية.

1. Introduction

The notion of fuzzy sets was presented by Zadeh [1], since that time, scientists and researchers have continued to develop fuzzy mathematical programming not only on a general level but also on a more practical level (see for instance [2, 3]. As well the concept of Rough sets was introduced by Pawlak [4], Youness [5] was the first who applythe Rough set theory to the single objective programming (SOP) problem and introduced a new optimization problem with a rough decision set and crisp objective function, called "rough single-objective programming" (RSOP) problem. He also presented the definition of two concepts for an optimal solution, namely "Surely Optimal" and "Possibly Optimal ".

Many attempts were made to overcome the concept of rough mathematical programming. For more information see [6, 7, 8] an interesting comparison between rough sets and fuzzy sets was given in [9].

Subsequently, for the aim of acquiring more realistic models and results of real-life Multiobjective programming (MOP) Atteya [10] presented a new extension of RSOP models presented by Osman in

[11] of the case of rough multiple objective programming (RMOP) problems.

Recently, many researchers have become interested in combining the concept of fuzziness with the concept of roughness in MOP problems, see [12, 13].

In this paper, we discuss the MONLP problem with rough objective functions and the fuzzy parameters in the constraints and introduce the algorithm for solving this type of MONLP problem.

2. Preliminaries

2.1 Rough Function

Definition 1 [14]. Let $\tilde{f}: R^n \to R$ and $r, \bar{r} \in R$ where $r > \bar{r}$, where the universal set of the functions is $U = \{f(x); f(x): R^n \to R\}$, and the set of functions $\{f_j\}$ is a lower approximation of $\bar{f}(x)$ which is denoted by $f^{\ell}(x)$ and defined as:

$$f^{\ell}(x) = \{f_j(v) \subset U: |f_j(x) - \tilde{f}(x)| < r\},$$

And the set of functions $\{f_k\} \subset U$ is an upper approximation of $\tilde{f}(x)$ which is denoted by $f^u(x)$ and defined as:

$$f^u(x) = \{f_x(x) \in U: |f_k(x) - \tilde{f}(x)| < \bar{r}\}.$$

where $\{f^{\ell}(x)\} \subseteq \{f^u(x)\}$. The function $\tilde{f}(x)$ is called a rough function if $f^{\ell}(x) \neq f^u(x)$.

Definition 2 [14]. The boundary region of the rough function $\tilde{f}(x)$ is equal to $f^u(x) - f^{\ell}(x)$, where $f^{\ell}(x)$ is the lower approximation of $\tilde{f}(x)$, and $f^u(x)$ is the upper approximation of $\tilde{f}(x)$.

2.2 Fuzzy Number

Definition 3 [15, 16]. A fuzzy subset $\hat{A}$ [of the real line $\mathbb{R}$ with membership function $\mu_{\hat{A}}: \mathbb{R} \to [0,1]$ is called a fuzzy number if:

(i) $\hat{A}$, is normal, i.e., there is $x_0 \in \mathbb{R}$ such that $\mu_{\hat{A}}(x_0) = 1$.

(ii) $\hat{A}$ is fuzzy convex that is: $\mu_{\hat{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_{\hat{A}}(x_1), \mu_{\hat{A}}(x_2))$,
for $x_1, x_2 \in \mathbb{R}, \lambda \in [0,1]$.

Definition 4: A trapezoidal fuzzy number $\hat{A}$ is denoted as $(a^L - \alpha, a^L, a^V, a^V + \beta)$. See figure1

a^L: the leftmost point where $\mu_{\hat{A}}(x)$ takes the value 1.
a^U: the rightmost point where $\mu_{\hat{A}}(x)$ takes the value 1.
α: the spread of the fuzzy number to the left of a^L.
β: the spread of the fuzzy number to the right of a^U.

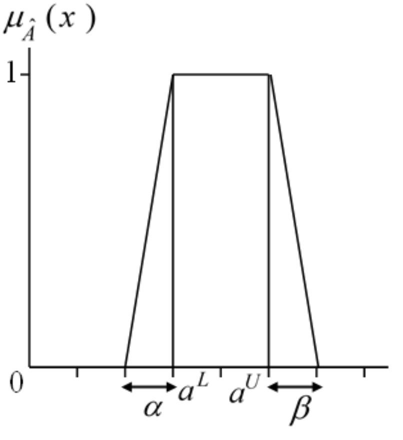

Figure1: A trapezoidal fuzzy number

2.3 Rouben's Ranking Function [17]

The ranking function suggested by F. Rouben [for the real numbers corresponding to the fuzzy numbers with respect to the function] defined by

$$\Re(\hat{a}) = \frac{1}{2}\left(a^L + a^U + \frac{1}{2}(\beta - \alpha)\right),$$

where $\hat{a} = (a^L - \alpha, a^L, a^U, a^U + \beta)$.

2.4. Weighted-Sum Method

A multiobjective problem is often solved by converting multiple objectives to a single-objective scalar function. This approach is in general known as the weighted sum of scalarization method. The weighting problem

$$P(w)\text{Min} \sum_{i=1}^{m} w_i f_i(x)$$

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subject to: $x \in X$,

where $W = \{w: w \in R^n, w_i \geq 0, i = 1,2, \ldots, m \text{and} \sum_{i=1}^{m} w_i = 1\}$ be the set of nonnegative weights.

Theorem 1 [18]. $\boldsymbol{x}^*$is a non-inferior solution of (VOP) vector optimization problem

if there exists $w \in W$ such that x^* solves $P(w)$ and if either one of the following two conditions holds:

(i) $w_i > 0$, for all $i = 1, \ldots, m$, or
(ii) x^* is the unique solution of $P(w)$.

Definition 5 [18]. x^* is said to be a non-inferior solution of VOP if there exists no other feasible x (i. e., $x \in X$) such that $f(x) \leq f(x^*)$, meaning that $f_i(x) \leq f_i(x^*)$ for all $i = 1,2, \ldots$,

3. Problem Formulation.

The general formulation for MONLP problems:

$$\text{Min}(f_1(x), f_2(x), \ldots, f_m(x))^T$$

subject to

$$x \in X = \{x \in \mathbb{R}^n: g_j(x) \leq 0, j = 1,2, \ldots, k\}, \quad (1)$$

where $f_i(x), i = 1,2, \ldots, m$ are objectives, $m \geq 2$, $g_j(x), j = 1,2, \ldots, k$ are constraints, and at least one i or j is a nonlinear function, $f_i(x), g_j(x)$ are real-valued functions.

In the reality of our daily life, the goals may not be completely clear to the owner of the problem or the decision maker (DM), so he may give us approximate goals that achieve his goal, that is, (s)he sets each unclear objective function between two objectives represent the lower approximation and upper approximation of the DM. This is in line with the definition of rough functions presented by Youness [14].

Also, the means that the knowledge available about the objective is not sufficient to describe the objectives accurately.

Thus, we here assume that the objective functions of MONLP problem are rough functions and the constraints contain fuzzy parameters (trapezoidal fuzzy numbers).
Therefore, the formula for the rough multiobjective nonlinear programming (RMONLP) problems with fuzzy parameters in constraints becomes the following:

$$\text{(RMONLP)} \qquad \min(\tilde{f}_1(x), \tilde{f}_2(x), \dots, \tilde{f}_m(x))^T$$
$$\text{subject to:}$$
$$x \in X = \{x \in \mathbb{R}^n : g_j(x, \hat{\lambda}) \le 0, j = 1,2,\dots,k\}, \qquad (2)$$

Where: $\tilde{f}_i(x), i = 1,2,\dots,m$ are rough objective functions, i. e., $f_i^{\ell}(x) \le \tilde{f}_i(x) \le f_i^{u}(x)$,
$g_j(x, \hat{\lambda}), j = 1,2,\dots,k$ are constraints with fuzzy parameters, $\hat{\lambda}$ is fuzzy parameter.

4. The Algorithm for Solving RMONLP Problem (2).

Step 1. Use the weighted sum method to make multiple objectives (RMONLP) a single-objective function as:

$$\text{Min} \sum_{i=1}^{m} w_i \tilde{f}_i(x)$$
$$\text{Subject to:}$$
$$x \in X = \{x \in \mathbb{R}^n : g_j(x, \hat{\lambda}), j = 1,2,\dots,k\}, \qquad (3)$$

where $\sum_{i=1}^{m} w_i = 1, f_i^{\ell}(x) \le \tilde{f}_i(x) \le f_i^{u}(x)$, $g_j(x, \hat{\lambda})$ are constraints with trapezoidal fuzzy numbers, $\hat{\lambda}$ is a fuzzy parameter.

Step 2. We deal with trapezoidal fuzzy parameters using the Rubens ranking function to get the real numbers corresponding to them

$$\Re(g_j(x, \hat{\lambda})) = g_j(x, \lambda); R(\hat{\lambda}) = \lambda, j = 1,2,\dots,k,$$

$g_j(x, \lambda)$ are constraints with corresponding real numbers.
Thus, the problem will become

$$\text{Min} \sum_{i=1}^{m} w_i \tilde{f}_i(x)$$

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subject to:

$$x = \{x \in \mathbb{R}^n : g_j(x,\lambda), j = 1,2,\dots,k\}, \qquad (4)$$

where: $f_i^{\ell}(x) \leq \tilde{f}_i(x) \leq f_i^{u}(x), g_j(x,\lambda)$ are constraints with real numbers corresponding to fuzzy parameters.

Step 3. To solve the previous problem (4) we need to minimize the boundary region of $\tilde{f}(x)$ that is, we need to solve the next boundary problem [14]:

$(BP)_w$

$$\min \mathrm{F}(x) = \sum_{i=1}^{m} w_i (f^{u}(x) - f^{\ell}(x))$$

subject to:

$$x \in X, \{x \in \mathbb{R}^n : g_j(x,\lambda) \leq 0, j = 1,2,\dots,k\},$$

are convex and continuous functions.

Hence, we can solve the last problem using KKT (Karush- Kuhn-Tucker) conditions.

Example 1: Consider the rough functions

$\tilde{f}_1 : M \to R$ with $f_1^{\ell} = x + y$, $f_1^{u} = \frac{1}{3}x^3 + \frac{7}{2}x^2 - 10y + 100$ and $\tilde{f}_2 : M \to R$ with $f_2^{\ell} = x - y^2$, $f_2^{u} = x^2 + y + 2$, and consider the following problem

$$min(\tilde{f}_1(x), \tilde{f}_2(x))^T ,$$

subject to:

$$x \in X = \{(x,y) \in R^2 | \hat{1}x + \hat{1}y \leq \hat{10}, \hat{3.5} \leq \hat{1}x \leq \hat{6}, \hat{1}y \leq \hat{6}, \hat{1}x - \hat{1}y \geq \tilde{1}\},$$

$$\Re(g(x,\hat{\lambda})) = g(x,\lambda),$$

$$\Re(\hat{a}) = \frac{1}{2}\left(a^L + a^U + \frac{1}{2}(\beta - \alpha)\right),$$

where

$$\hat{1} = (0.7,0.9,1.1,1.4), \hat{1} = (0.7,0.9,1.1,1.4),$$

$$\hat{10} = (9.7,9.9,10.2,10.5), \quad \hat{3.5} = (2.5,3.3,3.7,4.1),$$

$$\hat{1} = (0.8,0.9,1.1,1.5), \hat{6} = (5.5,5.8,6.1,6.6),$$

$\overset{\wedge}{1} = (0.8,1.1,1.3,1.6), \overset{\wedge}{6} = (5.4,5.9,6.3,6.8),$

$\overset{\wedge}{1} = (0.8,0.9,1.1,1.3), \overset{\wedge}{1} = (0.8,0.9,1.1,1.3)$

$\overset{\wedge}{1} = (0.8,0.9,1.1,1.3).$

They are arranged in accordance with the order mentioned in the constraints.

Step 1:

$$min(w_1\tilde{f}_1(x) + w_2\tilde{f}_2(x))$$

subject to:

$$\hat{1}x + \hat{1}y - \overset{\wedge}{10} \leq 0,$$
$$\overset{\wedge}{3.5} - \hat{1}x \leq 0,$$
$$\hat{1}x - \hat{6} \leq 0,$$
$$\hat{1}y - \hat{6} \leq 0,$$
$$\hat{1} - \hat{1}x + \hat{1}y \leq 0,$$
$$x, y \geq 0,$$

$$R(\hat{a}) = \frac{1}{2}\left(a^L + a^U + \frac{1}{2}(\beta - \alpha)\right)$$

Where:

$$\sum_{i=1}^{2} w_i = 1, w_i \geq 0,$$
$$x + y \leq \tilde{f}_1(x) \leq \frac{1}{3}x^3 + \frac{7}{2}x^2 - 10y + 100,$$
$$x - y^2 \leq \tilde{f}_i(x) \leq x^2 + y + 2,.$$

Step 2: After using the ranking function for constraints the problem becomes:

$$min \quad w_1\tilde{f}_1(x) + w_2\tilde{f}_2(x)$$

subject to

$$1.025x + 1.025y - 10.075 \leq 0,$$
$$1.075x - 6 \leq 0,$$
$$3.55 - 1.075x \leq 0,$$
$$1.2y - 6.1 \leq 0,$$
$$1 - x + y \leq 0,$$
$$x, y \geq 0,$$

$$\left(\text{i.e.,} R(\hat{1}) = \frac{1}{2}\left(0.9 + 1.1 + \frac{1}{2}(0.1)\right) = 1.025\right)$$

Note that in the next step we assume that $w_1 = 0.5, w_2 = 0.5$.

Step 3:

$$min(0.167x^3 + 2.25x^2 + 0.5y^2 - x - 5y - 49)$$

subject to

$$x + y - 9.829 \leq 0,$$
$$1.075x - 6 \leq 0,$$
$$3.55 - 1.075x \leq 0,$$
$$1.2y - 6.1 \leq 0,$$
$$1 - x + y \leq 0,$$
$$x, y \geq 0,$$

using KKT conditions we will find that the optimal solution for our problem at the point

$$\bar{x} = (5.4145, 4.4145),$$
$$min \quad F(\bar{x}) = 26.72861832697.$$

5. Conclusion.

We presented an algorithm for solving multiobjective nonlinear programming problems with rough objective functions and fuzzy parameters in constraints, the algorithm relied on three main steps.The first is the conversion of multiple objective functions into a single objective function.The second is the use of the concept of Rouben's ranking function to obtain the real numbers corresponding to the fuzzy parameters and thirdly deal with roughness objective function, consequently, the problem has become a crisp nonlinear programming problem that can be solved using KKT condition. An example was provided for illustration.

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