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SUBMONOIDS OF ABELIAN PARATOPOLOGICAL GROUPS

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Abstract

One of the important subclasses of abelian paratopological groups is called the free abelian paratopological group on a topological space. It was introduced in 2003 by Remaguara, Sanchis, and Tkachenko. In this paper, we introduce a single submonoid of the free abelian paratopological group on Alexandroff space, then we prove that this submonoid is a base at the identity element of the free topology of the abelian group. The main result of this paper is to give applications of this submonoid such as studying separation axioms, compactness, and other properties of the free topology on the abelian group.

Keywords: free group, Abelian paratopological group, free abelian paratopological group, Alexandroff space, neighborhood base at an identity.

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مونويدات جزئية من الزمر الباراتبولوجية الآبيلية

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ملخص

إحدى الزمر الفرعية المهمة من الزمر الباراتبولوجية الأبيلية تسمى الزمرة الباراتبولوجية الأبيلية الحرة على فضاء تبولوجي. لقد تم اثبات وجودها في عام 2003 بواسطة ريماجوارا وسانشيس وتكاتشينكو. في هذا البحث، قمنا بتشكيل مونويدة جزئية وحيدة من الزمرة الباراتبولوجية الآبيلية الحرة، ثم أثبتنا أن هذه المونويدة الجزئية هي قاعدة للتبولوجيا الحرة للزمرة الأبيلية عند عنصرها المحايد. النتائج الرئيسية لهذا البحث هي إيجاد تطبيقات على هذه المونويدة كدراسة مسلمات الفصل، التراص، وبعض الخصائص الأخرى للتبولوجي الحر على الزمرة الابيلية.

1. Introduction

Our main reference in the context of general topology is the book of Ryszard Engelking in [1]. We assume that the reader is familiar with the notions of free abelian groups, abelian paratopological groups, and free abelian paratopological groups on topological spaces. For more information on these notions see Remaguara, Sanshis, and Tkachenko in [2] and A. Elfard in [3], [4], and [5]. Submonoids play the significant rule on defining a base at the identity for the free abelian paratopological group *AP(X)* on an Alexandroff space *X*. A. Elfard in [5], introduced a base at the identity element of free paratopological groups on Alexandroff space *X*. In this paper, section 4, we provide new applications of this neighborhood base for the free topology of free abelian paratopological groups on Alexandroff space such as compactness, $T_0, T_1, and\ T_2$ separation axioms, and other properties on the free topology of the abelian group.

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2. Definitions and Preliminaries

A *paratopological group* is a pair (G, τ), where G is a group and τ is a topology on G such that the mapping $(x, y) \to xy$ of $G \times G$ into G is continuous. If the group G is abelian then (G, τ) is called abelian paratopological group.

If (G, τ) is abelian paratopological group, then simply we call it abelian paratopological group G.

The next proposition is in [3] and it describes a complete neighborhood base at the identity of any paratopological group as follows:

Proposition 2.1. *Let G be abelian group and let $\mathcal{N}$ be a collection of subsets of G, where each member of $\mathcal{N}$ contains the identity element 0 of G. Then the collection $\mathcal{N}$ is a base at 0 for a paratopological group topology on G if and only if the following conditions are satisfied:*

(1) *for all $U, V \in \mathcal{N}$, there exists $W \in \mathcal{N}$ such that $W \subseteq U \cap V$;*

(2) *for each $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ such that $V + V \subseteq U$;*

(3) *for each $U \in \mathcal{N}$ and $x \in U$, there exists $V \in \mathcal{N}$ such that $x + V \subseteq U$.*

Definition 2.2. [2] Let X be a subspace of a paratopological group G. Suppose that

(1) *the set X generates G algebraically, that is, $\langle X \rangle = G$:*

(2) *every continuous mapping $f: X \to H$ of X to an a paratopological group H extends to a continuous homomorphism $\hat{f}: G \to H$.*

Then G is called the free paratopological group on X and is denoted by $FP(X)$. By substituting “abelian paratopological group” for each occurrence of “paratopological group” above we obtain the definition of the free abelian paratopological group on X, and we denote it by $AP(X)$.

We denote the free topology of $AP(X)$ by $\mathcal{g}$ and we note that the topology $\mathcal{g}$ is the strongest paratopological group topology on the underlying set $A_a(X)$ of $AP(X)$, that induces the original topology on X.

Let G be a group and let H be a subset of G. Then we say that H is a *submonoid* of G if H contains the identity of G and equipped with the associative binary operation of G.

Definition 2.3. [6] *Let X be a topological space. For each $x \in X$, we define the set $U(x) = \bigcap\{U : x \in U \text{ and } U \text{ is open}\}$. A space X for which every $x \in X$, the set $U(x)$ is open is called an Alexandroff space.*

Note that $U(x)$ is the smallest open subset of X containing x.

3. Bases at the identity element of AP(X)

The next theorem is proved by Elfard in [5].

Theorem 3.1. *The free abelian paratopological group $AP(X)$ on a space X is Alexandroff if and only if the space X is Alexandroff.*

Proposition 3.2. *If G is an abelian group and H is a submonoid of G, then $\{H\}$ is a neighborhood base at the identity element 0 of G for a paratopological group topology Ton G.*

Proof. We must verify that $\{H\}$ satisfies the three axioms of Proposition 2.1. If $U, V \in \{H\}$ where $U = V = H$, then there exists $W = U$ such that $W \subseteq U \cap V$. If $U \in \{H\}$, there exists $V = U = H \in \{H\}$ such that $2V \subseteq U$. If $U = H \in \{H\}$ and $g \in U$, then there exists $V = H \in \{H\}$ such that $g + V \subseteq U$. Therefore, $\{H\}$ is a base at 0 for a paratopological group topology T on G.

Let $W = \bigcup_{x \varepsilon X}(U(x) - x) \subseteq AP(X)$, where $U(x)$ as defined above. Then we define M to be the smallest submonoid of $AP(X)$ containing the set W. Therefore, M is of the form:

$$M = \{y_1 - x_1 + y_2 - x_2 + \cdots + y_n - x_n : x_i \in X, y_i \in U(x_i) for\ all\ i = 1,2,\dots,n, \quad n \in N\}$$

Let $\mathscr{M} = \{M\}$. Since M is a submonoid of $AP(X)$, by Proposition 3.2, $\mathscr{M}$ is a neighborhood base at the identity 0 of $AP(X)$ for a paratopological group topology T on the underling set $A_a(X)$ of $AP(X)$.

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We now give an important result with complete proof for the topology T on $AP(X)$.

Proposition 3.3. *The topology T induces a topology coarser than the topology on X.*

Proof. Let $y \in X$. We show that $(M + y) \cap X \subseteq U(y)$. Let $w \in M$. Then there exists $m \in \mathbb{N}$ such that $w = u_1 - x_1 + u_2 - x_2 + ... + u_m - x_m$, $x_i \in X$, $u_i \in U(x_i)$ for all $i = 1, 2, ..., m$. Here if $w + y \in (M + y) \cap X$, there exists $z \in X$ such that:

$$u_1 - x_1 + u_2 - x_2 + \cdots + u_m - x_m + y - z = 0 \qquad (1)$$

We need to show that $z \in U(y)$. From (1), If $z = y$ then $z \in U(y)$. If $z \neq y$ then y must equal x_i for some $i = 1, 2, ..., m$, say $y = x_1$. Hence, $U(x_1) = U(y)$. Also, $z = u_i$ for some $i = 1, 2, ..., m$. Without loss of generality, we can assume that $z = u_2$. Thus, $z \in U(x_2)$. Hence equation
(1) becomes:

$$u_1 - x_2 + u_3 - x_3 + \cdots + u_{n-1} - x_{n-1} + u_m - x_m = 0 \qquad (2)$$

If $u_1 = x_2$ then $z \in U(x_2) \subseteq U(x_1) = U(x_y)$. If $u_1 \neq x_2$ then let $u_1 = x_{i_k}$ and $x_2 = u_{i_1}$ for some $i_1, i_k \in \{3, 4, ..., m\}$. Either $k = 1$, then $z \in U(x_2) \subseteq U(x_{i_1}) \subseteq U(x_1) = U(y)$ or $k \neq 1$, then assume that $x_{i_1} = u_{i_2}$. Either $k = 2$, then $z \in U(x_2) \subseteq U(x_{i_1}) \subseteq U(x_{i_2}) \subseteq U(x_1) = U(y)$ or $k \neq 2$, then we take another step. This process must be stop because of the finite number of x_i's and u_i's and finally we get $z \in U(x_2) \subseteq U(x_{i_1}) \subseteq U(x_{i_2}) \subseteq ... \subseteq U(x_{i_{m-3}}) \subseteq U(x_{i_{m-2}}) \subseteq U(x_1) = U(y)$. Therefore, $z \in U(y)$ and then we get $(M + y) \cap X \subseteq U(y)$ which means $T \cap X \subseteq \tau$.

If $u_1 = x_2$ then $z \in U(x_2) \subseteq U(x_1) = U(x_y)$. If $u_1 \neq x_2$ then let $u_1 = x_{i_k}$ and $x_2 = u_{i_1}$ for some $i_1, i_k \in \{3, 4, ..., m\}$.

Either $k = 1$, then $z \in U(x_2) \subseteq U(x_{i_1}) \subseteq U(x_1) = U(y)$ or $k \neq 1$, then assume that $x_{i_1} = u_{i_2}$. Either $k = 2$, then $z \in U(x_2) \subseteq U$

$(x_{i_1}) \subseteq U(x_{i_2}) \subseteq U(x_1) = U(y)$ or $k\neq 2$, then we take another step. This process must be stop because of the finite number of x_i's and u_i's and finally we get $z \in U(x_2) \subseteq U(x_{i_1}) \subseteq U(x_{i_2}) \subseteq \ldots \subseteq U(x_{i_{m-3}}) \subseteq U(x_{i_{m-2}}) \subseteq U(x_1) = U(y)$. Therefore, $z \in U(y)$ and then we get $(M + y) \cap X \subseteq U(y)$ which means $T \cap X \subseteq \tau$.

The next theorem describes a base at the identity element 0 of the free abelian paratopological group $AP(X)$ on Alexandroff space X.

Theorem 3.4. *The collection $\mathscr{M}$ is a neighborhood base at the identity element* 0 *of $AP(X)$ for the free topology $\mathscr{g}$ of $AP(X)$.*

Proof. We show first that the topology T induced by $\mathscr{M}$ is finer than the free topology $\mathscr{g}$ of $AP(X)$. Let $\mu: X\rightarrow G$ be a continuous mapping of the space X into an abelian paratopological group G. Then μ extends to a homomorphism $\hat{\mu}: A_a(X)\rightarrow G$. We show that $\hat{\mu}$ is continuous concerning the topology T. Let V be a neighborhood of $\hat{\mu}(0) = 0$ in G and fix $x \in X$. Then $\mu(x) + V$ is a neighborhood of $\mu(x)$ in G. Since μ is continuous at the point x, $\mu(U(x)) \subseteq (\mu(x)+V)$ and since $\hat{\mu}|X = \mu$, $\hat{\mu}(U(x)) \dot\subseteq (\hat{\mu}(x) + V)$ which implies that $\hat{\mu}(U(x) - x) \dot\subseteq V$. Since x is any point in X, $\hat{\mu}(\bigcup_{x\varepsilon X}(U(x) - x)) \subseteq V$ Fix $n \in N$. Then there exists a neighborhood U of 0 in G such that $nU \subseteq V$. Since V is an arbitrary neighborhood of 0 in G, $\hat{\mu}(\bigcup_{x\varepsilon X}(U(x) - x))\subseteq U$ and then $n(\hat{\mu}(\bigcup_{x\varepsilon X}(U(x) - x)) \subseteq nU \subseteq V$. So

$$\hat{\mu}\left(n\left(U_{x\epsilon X}(U(x)-x)\right)\right) \dot\subseteq V \qquad (1)$$

Since (1) holds for each $n \in N$, $\hat{\mu}(\bigcup_{n\varepsilon N} n(\bigcup_{x\varepsilon X}(U(x) - x)))\subseteq V$. Since $M = \bigcup_{n\varepsilon N} n(\bigcup_{x\varepsilon X}(U(x) - x)), \hat{\mu}(M)\subseteq V$. Thus, $\hat{\mu}$ is continuous for the topology T, so that the topology T is finer than the free topology $\mathscr{g}$ of $AP(X)$. By Proposition 3.3, $T|_X$ is coarser than the original topology on X and since T is finer than the free topology $\mathscr{g}$, $T|_X$ is the original topology on X. Thus, we satisfied the conditions of Definition 2.2, which implies that $T = \mathscr{g}$. Therefore, $\mathscr{M}$ is a neighborhood base at 0 for the free abelian paratopological group $AP(X)$.

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4. Applications on the collection $\mathscr{M}$

Let X be Alexandroff space and let $AP(X)$ be the free abelian paratopological group on X. We define $\mathscr{B}= \{M + g: g \in A_a(X)$ *to* be a collection of subsets of $AP(X)$.

Lemma 4.1. *Let $B_1, B_2 \in \mathscr{B}$, where $B_1 = M + g_1$ and $B_2 = M + g_2$, $g_1, g_2 \in A_a(X)$. If $g_1 \in B_2$, then $B_1 \subseteq B_2$.*

Proof. Let $g_1 \in B_2$, then there is $w_1 \in M$, such that $g_1 = w_1 + g_2$. Let $h \in B_1$, thus there is $w_2 \in M$ such that $h = w_2 + g_1$. Because M is a submonoid of $A_a(X)$, hence $h = w_2 + w_1 + g_2 \in B_2$. Therefore, $B_1 \subseteq B_2$.

Proposition 4.2. [6] *Let X be an Alexandroff space and $\mathcal{U}$ be a family of open subsets of X, then $\mathcal{U}$ is the minimal base for the topology of X if and only if:*

(1) $\mathcal{U}$ covers X;

(2) if $U_1, U_2 \in \mathcal{U}$, there exists a subfamily $\{U_i: i \in I\}$ of $\mathcal{U}$ such that $U_1 \cap U_2 = \bigcup_{i\epsilon I} U_i$;

(3) if a subfamily $\{U_i : i \in I\}$ of $\mathcal{U}$ satisfies $\bigcup_{i\epsilon I} U_i; \in \mathcal{U}$, then there exists $i_0 \in I$ such that $\bigcup_{i\epsilon I} U_i; = U_{i_0}$.

By using the last proposition, we now provide the following result.

Theorem 4.3. *The base $\mathscr{B}$ as defined above is the minimal base for the free topology $\mathscr{g}$ of $AP(X)$.*

Proof. We show that the base $\mathscr{B}$ satisfies the conditions of Proposition 4.2. It is clear that $\mathscr{B}$ covers $A_a(X)$ and then satisfies condition (1). To show that $\mathscr{B}$ satisfies condition (2). Let B_1, $B_2 \in \mathscr{B}$ where $B_1 = M + g_1$ and $B_2 = M + g_2$, and $g_1, g_2 \in A_a(X)$. To show that, $B_1 \cap B_2 = \bigcup\{M + g,$ for all $g \in B_1 \cap B_2\}$. Let $h \in B_1 \cap B_2$, to show that, $h \in \bigcup\{M + g,$ for all $g \in B_1 \cap B_2\}$, we must show that $h \in (M + g^*)$ for some $g^* \in B_1 \cap B_2$. Suppose not. So, $h \notin (M + g)$ for all $g \in B_1 \cap B_2$. From last Lemma, we get $h \notin M + g$ for all $(M + g) \subseteq B_1 \cap B_2$. This is a contradiction, because $\mathscr{B}$ is a base for the

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topology $\mathcal{g}$ of $AP(X)$. Thus, $h \in (M + g^*)$ for some $g^* \in B_1 \cap B_2$. Hence $g \in \bigcup\{M + g, \text{for all } g \in B_1 \cap B_2\}$. So, we have:

$$B_1 \cap B_2 \subseteq U\{M + g. for\ all\ g \in B_1 \cap B2 \qquad (1)$$

Now, for all $g \in B_1 \cap B_2$, by last Lemma, $M + g \subseteq B_1 \cap B_2$. Hence $U\{M + g. for\ all\ g \in B_1 \cap B_2\} \subseteq B_1 \cap B_2 \qquad (2)$

Therefore, from (1) and (2) we get the required result.

To show that $\mathcal{B}$ satisfies condition (3). If a subfamily $\{M + g_i : g_i \in A_a(X), i \in I\}$ of $\mathcal{B}$ satisfies $\bigcup_{i\epsilon I}\{M + g_i\} \in \mathcal{B}$, then there exists $i_0 \in I$ such that $\bigcup_{i\epsilon I}\{M + g_i\} = M + g_{i_0}$. Suppose that $\bigcup_{i\epsilon I}\{M + g_i\} = M + g*$ for some $g^* \epsilon A_a(X)$. We need to show that $M+g^* = M + g_{i_0}$ for some $i_0 \epsilon I$. Let $g \epsilon M + g*$, $then\ from\ our\ assumption\ g \epsilon \bigcup_{i\epsilon I}\{M + g_i\}$. Thus there exists $i_0 \epsilon I$ such that $g \epsilon M + g_{i_0}$. Hence $M+g^* \subseteq M + g_{i_0}$. Since $g_{i_0} \epsilon \bigcup_{i\epsilon I}\{M + g_i\}$, $i_0 \in I$, this implies that $g_{i_0} \epsilon M + g*$ and then $M + g_{i_0} \subseteq M + g*. Therefore, M + g* = M + g_{i_0}$ and then $\mathcal{B}$ is the minimal base for the topology of $AP(X)$.

Corollary 4.4. *If X is an Alexandroff space then for all $B \in \mathcal{B}$, B is a compact subset of $AP(X)$.*

Proof. By Theorem 3.1, $AP(X)$ is Alexandroff. Let $B \in \mathcal{B}$, where $B = M + g$, $g \in A_a(X)$. Let $\{U_i\}_{i\epsilon I}$ be an open cover on B. Then $g \in U_{i_0}$ for some $i_0 \in I$. Thus $B = M + g \subseteq U_{i_0}$ which means that U_{i_0} is a finite subcover of $\{U_i\}_{i\epsilon I}$. Therefore, B is compact.

We now provide a condition for the base $\mathcal{B}$ to be the free abelian paratopological group $AP(X)$ on an Alexandroff space X is a T_0-space.

Theorem 4.5. *The free abelain paratopological group $AP(X)$ on an Alexandroff space X is a T_0-space if and only if whenever $M + g_1 = M + g_2$, where $g_1, g_2 \in A_a(X)$ we have $g_1 = g_2$.*

Proof. *(⇒):* Assume that $AP(X)$ is a T_0-space and let $g_1, g_2 \in AP(X)$ where $g_1 \neq g_2$. Since $AP(X)$ is a T_0-space, so we can find an element $g \in A_a(X)$ such that $g_1 \in M + g$ and $g_2 \notin M + g$ or $g_2 \in M + g$ and $g_1 \notin M + g$. If $g_1 \in M + g$ and $g_2 \notin M + g$, then $M + g_1 \subseteq M + g$ and $M + g_2 \nsubseteq M + g$. Therefore, $M + g_1 \neq M + g_2$.

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(⇐): Let $g_1, g_2 \in A_a(X)$ and $g_1 \neq g_2$, then $M + g_1 \neq M + g_2$. Hence, $M + g_1 \nsubseteq M + g_2$ or $M + g_2 \nsubseteq M + g_1$. If $M + g_1 \nsubseteq M + g_2$, then we get $g_1 \notin M + g_2$. Therefore $AP(X)$ is a T_0-space.

Note that the free abelian paratopological group $AP(X)$ on Alexandroff space X is T_1 if and only if $M + g = \{g\}$. It follows that $AP(X)$ is discrete and so every element of $A(X)$ is an isolated element.

Proposition 4.6. *If the free abelian paratopological group $AP(X)$ is Housdorff Alexandroff then for all $g_1, g_2 \in A_a(X)$ where $g_1 \neq g_2$ we have $M + g_1 \cap M + g_2 = \phi$.*

Proof. Let $g_1, g_2 \in A_a(X)$ where $g_1 \neq g_2$. If $AP(X)$ is Hausdorff there exist tow disjoint open sets U, V of $AP(X)$ such that $g_1 \in U, g_2 \in V$. Because $AP(X)$ is Alexandroff, so $M + g_1 \subseteq U, M + g_2 \subseteq V$. Therefore $M + g_1 \cap M + g_2 = \phi$.

Proposition 4.7. *The free abelian paratopological group $AP(X)$ is a Hausdorff Alexandroff if and only if $AP(X)$ is discreet.*

Proof. (⟹): Assume that AP(X) is a Hausdorff Alexandroff. To show that $M + g = \{g\}$ for all $g \in AP(X)$. Let $w \in M + g$. Then $M + w \subseteq M + g$ which means $M + w \cap M + g = M + w$. Since $M + w \neq \phi$ and $AP(X)$ is Hausdorff thus $w = g$ and $\{g\}$ is open. Therefore $AP(X)$ is discrete.

(⟸): Assume that $AP(X)$ is discrete. So, it is Hausdorff and because every subset of $AP(X)$ is open then $AP(X)$ is Alexandroff.

Theorem 4.8. *Let $f: AP(X) \longrightarrow AP(Y)$ be an open and onto continuous function. If $AP(X)$ is Alexandroff then $AP(Y)$ is Alexandroff.*

Proof. Let $g \in AP(Y)$ then there exists $w \in AP(X)$ such that $f(w) = g$. Because f is open then $f(M + w)$ is open in $AP(Y)$. Let U be an open set in $AP(Y)$ where $g \in U$. thus $w \in f^{-1}(U)$. Since $f^{-1}(U)$ is open in $AP(X)$, so $M + w \subseteq f^{-1}(U)$. Therefore $f(M + w) \subseteq U$ and then $AP(Y)$ is Alexandroff with $M + g = f(M + w)$.

Corollary 4.9. *Let $f : AP(X) \rightarrow AP(Y)$ be a homeomorphisim function. If $AP(X)$ is Alexandroff then $AP(Y)$ is Alexandroff.*

5. Conclusion

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In this paper, we introduce a single submonoid of the free abelain paratopological group on Alexandroff space and then in Theorem 3.4, we prove that this submonoid is a base at the identity element of the free topology of the abelian group. The main results of this paper are Theorem 4.3, Theorem 4.5, and Proposition 4.7.

Recommendations. Elfard in [5] established a new subclass of free paratopological groups which is called the Alexandroff free paratopological groups and he constructed a base for its free topology. In this paper, we introduce new applications on this base. For our recommendations, it is important to study other free topological properties on this subclass such as, connectedness, separation axioms, countability axioms.

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